# Diamond Relations

*Sketch of a theory of diamond relations*

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## Abstract

Because of their concreteness, the complexity of *relations* is more structured and is not always tackled by the axioms or properties of mathematical categories. E.g. the categorical properties of *commutativity* and *transitivity* are not necessarily holding for relations.
As an application, relations and the category of PATH as proposed by Pfalzgraf is presented. Diamond relations and a diamond version of PATH, i.e. JOURN (journey), based on diamond set theory, is sketched.
How to introduce intransitivity (non-commutativity) in category theory? Two approaches are presented: Pfalzgraf's *generalized* morphisms which are re-establishing categorical commutativity on a generalized level of relations and a sketch of polycontextural diamond constructions which are introducing different types of non-commutativity on the level of a generalized (disseminated) paradigm of categoricity.

# 1. Diamond relations

## 1.1. Set-theoretical relations

### 1.1.1. Composition of relations

*A relation* from set $A_0$ to *a* set $A_1$ is *a* triple
$(A_0, R, A_1)$ with $R \subseteq A_0 \times A_1$,
its *composition* with $(B_0, R, B_1)$ is defined iff $A_1 = B_0$.

The composition of two relations $R: A \longrightarrow B$ and $S: B \longrightarrow C$ is given by:
$(a, c) \in S \circ R$ iff for some $b \in B$, $(a, b) \in R$ and $(b, c) \in S$.
That is: $(a, c) \in S \circ R \Longleftrightarrow \exists\, b \in B : (a, b) \in R \wedge (b, c) \in S$.

Obviously, sets like A, B and C with their elements a, b and c belong to the universe of sets U: A, B, C ⊂ U.

### 1.1.2. Composition of morphisms

"A binary operation o, called *composition* of morphisms, such that for any three objects a, b, and c, we have hom(a, b) × hom(b, c) -> hom(a, c).
The composition of f: a -> b and g: b -> c is written as go f or gf (some authors write fg), governed by two axioms: *Associativity* and *Identity*." (WiKi)

Mostly, the first-orderr logic and set theoretical notions to build category theoretical constructions, are not formalized in the sense of formal logic and set theory.

$A \xrightarrow{f} B$ o $B \xrightarrow{g} C$ => $A \xrightarrow{gf} C$, with cod(f) = dom(g)

Diagram
A-->B
↘ ↓
C commutes.

### *Category theory as a first-order theory*

From: William S. Hatcher (The Foundation of Mathematics, 1968)

$(C1)\,(D(C(x)) = C(x) \wedge C(D(x)) = D(x)$

$(C2)\,\forall x \forall y \forall z \forall u (K(x, y, z) \wedge K(y, z, u) \supset z = u)$

$(C3)\,\forall x \forall y ((Ez: K(x, y, z) \equiv (C(x) = D(y)))$

$(C4)\,\forall x \forall y \forall z (K(x, y, z) \supset (D(z) = D(x) \wedge C(z) = C(y)))$

$(C5)\,\forall x (K(D(x), x, x) \wedge K(x, C(x), x))$

(C6) Associativity.

http://www.thinkartlab.com/pkl/lola/graphematische_problem-kae.pdf

### 1.1.3. Transitivity of relations

$\forall a, b, c: aRb \wedge bRc \Longrightarrow aRc$

"In mathematics, a binary *relation* R over a set X is transitive if whenever an element a is related to an element *b*, and *b* is in turn related to an element *c*, then *a* is also related to *c*." (WiKi)

### 1.1.4. Intransitivity of relations

$\neg \forall a, b, c: aRb \wedge bRc \Longrightarrow aRc$ : Intransitivity

$\forall a, b, c: aRb \wedge bRc \Longrightarrow \neg aRc$ : Antitransitivity

### 1.1.5. Intransitivity for composition

$\forall a, b, c: \hom(a, b) \times \hom(b, c) \longrightarrow \neg \hom(a, c)$

$\neg \forall a, b, c: \hom(a, b) \times \hom(b, c) \longrightarrow \hom(a, c)$

$\forall a, b, c: a \longrightarrow b \circ b \longrightarrow c \Longrightarrow \neg (a \longrightarrow c)$

$\neg \forall a, b, c: a \longrightarrow b \circ b \longrightarrow c \Longrightarrow a \longrightarrow c$

Diagram
A⟶B
¬ ↘ ↓
C does **not** commute.

Does this construction make any sense for category theory? Obviously not. It doesn't accept the main definition of categorical composition of morphism.

## 1.2. Diamond theory of relations

### 1.2.1. Diamond relation

Diamond relations are defined over the pluri – verse of acceptional and rejectional sets.

Classical (acceptional) binary relation: $R \subset (X \circ X)$

Non – classical (rejectional) binary relation : $r \subset (x \bullet x)$

Diamond binary relation: $(R,\ r) \subset \subset ((X,\ x) \circ\bullet (X,\ x))$.

**Diamond relation**

$$(R,\ r) \subset \subset [(X,\ x) \diamond (X,\ x)] \text{ iff } \begin{bmatrix} R \subset (X \circ X) \\ r \subset (x \bullet x) \end{bmatrix}$$

For each *binary* relation $R \subset X \circ X$, with $X \in U$, there is complementary *unary* relation $r \subset (x,\ x)$ with element $x \in \overline{U}$.

For each *ternary* relation $R \subset X \times X \times X$, with $X \in U$ there is a complementary *binary* relation $r \subset (x,\ x)$ with $x \in \overline{U}$.

**Diamond relation DiamRel:**

$R \in Cat,\ r \in Sat$

$$(R,\ r)^{(m)} \Longleftrightarrow Rel^{(m)} \,\|\, rel^{(m-1)}$$

## 1.2.2. Diamond composition of relations

**Diamond Composition of Relations**

A relation from tupel $(A_o^1, A_o^2)$ to a tupel $(A_1^1, A_1^2)$ is a 2 – triple $((A_o^1, A_o^2),\ R^{1.2},\ (A_1^1, A_1^2))$ with $R^{1.2} \subseteq (A_o^1, A_o^2) \times\times (A_1^1, A_1^2)$,

its composition with $((B_o^1, B_o^2)\ R^{1.2}\ (B_1^1,\ B_1^2))$

$$\text{is diamond – defined iff} \begin{bmatrix} \begin{bmatrix} \delta(A_1^2) & \doteq \delta(B_1^2) \\ \delta(\alpha_2) & \doteq \alpha_4 \\ \delta(\omega_1) & \doteq \omega_4 \end{bmatrix}_{DIAM} \\ \begin{bmatrix} \omega_1 & \simeq & \alpha_2 \\ A_1^1 & \equiv & B_o^1 \end{bmatrix}_{REL} \end{bmatrix}$$

$\left(\left(A_o^1, \alpha_1\right), R^{1.2}, \left(A_1^1, \omega_1^1\right)\right)$ with $R^{1.2} \subseteq \left(A_o^1, \alpha_2\right) \times\times \left(A_1^1, \omega_2\right)$,

is a composition with $\begin{bmatrix} \left(\left(B_o^1, \alpha_3\right), R^{1.2}, \left(B_1^1, \omega_3\right)\right) \\ \left(\left(B_o^1, \alpha_4\right), \overline{R^{1.2}}, \left(A_1^1, \omega_4\right)\right) \end{bmatrix}$

Diamond – Relation composition :

$\left(a, b\right) \in S \Diamond R \iff \left(\frac{S \circ R}{\overline{S \circ R}}\right) =$

$S \circ R$: $\exists\, b = \left(b_1^1 \equiv b_0^2\right) \in B : \left(a_0^1, b_1^1\right) \in R^1 \wedge \left(b_0^2, c_1^2\right) \in S^2$

$\_$: $\left(a^4, b^4\right) \in \overline{S \circ R} \iff \exists\, b_1^1 \in B \wedge \exists\, b_0^2 \in B$: $\mathrm{diff}\left(b_1^1, b_0^2\right) \in R^4$

Short:

**Binary diamond composition**

$\forall A \forall B \in U, \forall D \in \overline{U}, U \frown \overline{U} = \emptyset,$

$\left(a, b\right) \in S \Diamond R \iff \exists\, b \in B \,\|\, \exists\, d \in D:$

$\left(a, b\right) \in R \wedge \left(b, c\right) \in S \,\|\, \left(d\right) \in \overline{S \circ R}$

$\left(a, b\right) \in S \Diamond R \iff \exists\, b \in B:$

$\left(a, b\right) \in R \wedge \left(b, c\right) \in S \,\|\, \exists\, d \in D: \left(d\right) \in \overline{S \circ R}$

1. A. A. Mullin: *Properties of mutants.*

Let $(A, *)$ denote a nonempty set $A$ together with a closed binary composition law "$*$" defined on $A$. By a *mutant* of $(A, *)$ is meant a subset $M$ of $A$ that satisfies the condition that $M * M \subseteq \overline{M}$, where $M * M = \{a * b : a \in M \text{ and } b \in M\}$ and $\overline{M}$ is the set of all of the elements of $A$ not in $M$. If all of the elements of $A$ are idempotent with respect to "$*$" let the empty set be the only mutant of $(A, *)$.

# 2. Category PATH

## 2.1. Motivation

### *Diamond motivation*

From the point-view of diamonds, the question about *non*-identitive, *non*-commutative and *non*-transitive compositions for diamonds might arise. Non-transitivity is a well known property for so called "real life" situations. Again, a first observation to remind, is the fact, that in contrast to classical logic, the operator "non" has many well-defined appearances in polycontextural logics.
Non-transitivity in diamond theories, thus, is not simply a total negation or rejection of transitivity but the acceptance of a plurality of different kinds of transitivity, enabling many kind of specific non-transitive relations.
Nontransitivity appears naturally for relations. Categories are by definition transitive (commutative). Hence, intransitivity for categories can be introduced only as a secondary concept. On the other hand, intransitivity for relations might be transformed to transitivity by a kind of a generalization or an abstraction to generalized relations, i.e. *"a more general type of morphism"* based on the difference of *direct* and *indirect* arrows (Pfalzgraf).

It is based on a very different paradigm to ask: *"How to introduce intransitivity on the epistemological level of the definition of categories as such?"*

It shall be shown, say sketched, that such a basic interplay of transitivity and different forms of non-transitivity is accessible in the framework of a polycontextural diamond category theory.

**Road Map Metaphor**

"Let us consider, for illustration, a simple practical example of *real* life: Looking at general relational structures is quite natural since transitivity and even reflexivity are not always existent in *applications*.
As a practical example let us look at a *road map* where the nodes (objects) are towns and the arcs (arrows) are road connections, then not every pair of towns has a *direct* connection (arrow), in general. Therefore, generally, starting from a point we have to follow a path of direct road connections passing several nodes (towns) before we can reach a goal." (Pfalzgraf) [my emph]

Pfalzgraf gives an example about direct connections between towns. The same observation holds for most intensional verbs, like *win*, *love*, *hate*, etc., e.g. *A loves B*, *B loves C.* Does *A loves C* hold necessarily? Obviously not.

Pfalzgraf's strategy to keep transitivity by *generalization* could be paraphrased as:
*A loves B*, *B loves C*, *A hate C* , then, by generalization from intransitivity to transitivity:
*A is-in-emo-relation to B*,
*B is-in-emo-relation to C*, hence,
*A is-in- emo-relation to C* .

On the other hand, if A is connected with B, and B is connected with C, then A is connected with C, too. At least in a stable world, where the definition of connection is not suddenly transforming itself.
But is this reflection on intransitivity matching the level of categorical abstractions, like morphisms, compositions, matching conditions? Obviously not!

Such examples are localized on the level of *relational* algebra and logic, i.e. based on *set* theoretical and not on *category* theoretical assumptions.

Pfalzgraf gives a relational definition of situations where transitivity and reflexivity doesn't hold. Then he extends his *"CAT modeling approach"* to *"contain arbitrary relations"*. On the base of this relational concept he abstracts the category **PATH**.

Transitivity gets re-established with Pfalzgraf's approach by the more general abstraction of *"consecutive arrows"* in contrast to *"direct arrows"*. In the example below, "x-->z" is a direct arrow (morphism) which is not covered in the relational approach, but the *consecutiveness* of arrows "x -->y -->z" as such is considered as a *generalized* morphism Mor(x,z).

The proposed "*extension*" of categories, based on the distinction "direct" versus "consecutive" arrows (morphisms), towards a category of relations seems to be an *application* of "pure" category theory to the theory of relations and not in itself a categorical extension, but an application. At the end, all gets saved, back home in the category PATH.

**Activity Map Metaphor**
Instead of connecting towns, i.e. objects, the diamond approach is focusing on connection *activities*, i.e. arrival/departure-activities, of journeys between towns. The set- and object-aspect of towns is secondary to the activities of *coming (arriving)* and *going (departing)*. It is just this focussing on *activities* which is enabling the discovery of the concept of hetero-morphism and the construction of saltatories that are complementary, and not simply dual, to categories.

**First- and second-order thematizations.**
Now, thinking seems to be an activity too. Hence, the process of thinking might be connected, step by step. Steps are represented by morphisms, morphisms are informational, information is based on first-order observation. The process of composition isn't in focus. In focus are morphisms based on objects. But objects are moved into the background in favor of morphisms. Therefore it is strict forward to think of non-transitive connections of morphisms as a morphism. This is a kind of an abstraction towards a "more general" concept of morphism. This abstraction is abstracting in two directions, one from the objects and one from the composition of the less general, i.e. ordinary morphisms.

*"We point out here that in our definition of agents human agents are included."* (Pfalzgraf, 07, p.33)

On this level of abstraction it is not only generous but necessary to include human actors together with any other actors into the general approach of MAS. As far as we reduce human actions to non-reflectional roles, like using a mobile phone, e-Business, this reduction is appropriate. But as usual, the claims are much higher.

If thinking the process of thinking is understood as something different to the mere process of thinking (of something) then, such a second-order thinking is not properly modeled by morphisms but by a reflection on the conditions of morphisms which are, in this scenario, compositions, the operation to compose morphisms. The alter of morphisms are compositions. But that is demanding for a radically new abstraction. An abstraction which is not objectivizing compositions to morphisms of ordinary or general abstraction, but is keeping the processuality of the process in its own right and domain.

The argumentative figure to this difference is this: classic thinking is thinking of something. Therefore, for classical thinking, second-order thinking is thinking of thinking of something, i.e. thinking of something and this something is a “thinking of something”, hence, still accessible to thinking as thinking of something - and nothing else. As a result, classical thinking is inevitable connected with the problem of infinite regress of its meta-levels (type theory, meta-circularity).

Transclassical thinking insists on the structural difference between thinking of *something* and thinking of *thinking* as the process of thinking (of something). It is the processuality which isn’t caught by the ontology of something, even if this something is not stubbornly thought as an identical object but as a temporal object or process (Whitehead).
If there will be, one day, something like a Semantic Web, it will not be based on morphisms or on generalized morphisms.

Second-order thinking or, as it is called more properly, transcendental logic (Kant, Husserl, Gunther), is not formalized by category theory or relational algebra and logic.

The only account I’m aware of is the approach I proposed myself as diamond category theory. Diamond theory takes thinking a step further than polycontextural logic as it was introduced by Gunther as a first step towards a formalization of dialectics and transcendental logic and as it was further developed in the last decades by my own work.

As the transcendental logical tradition (Kant, Fichte, Schelling) pointed out, thinking is doubled. Fichte calls it the *“Duplizität des Ich”*. The process of thinking, which is thinking of thinking, is parallactic and antidromic. This exactly is what diamond theory discovered. The crucial difference is that diamond theory is formal and operative, albeit in a new sense of the terms but nevertheless well connect to the tradition of mathematical thinking, and not speculative descriptive like the transcendental logic forerunner.

**Diamond non-transitivity**
What could be a reasonable extension of category theoretical definitions *per se* introduced from a diamond approach? The strategy, that has to be *excluded* first, is to go back to application, concretization and other methods, like fiberization, which are, despite their productivity, leaving the level of category theoretical abstractions.
The only chances I see at the time for a structural enhancement of categorical notions, beyond fibered and n-categories, seems to be sketched by the diamond approach to compositionality of morphisms, i.e. the complementarity of categories and saltatories.

### *Categorical motivation*

Following Jochen Pfalzgraf

"For practical reasons - in order to reach a large area of applications - we extend our CAT modeling approach to *arbitrary relations* (X,R). In such a case, we are not able to associate directly a category to the relation as we did it before since transitivity, reflexivity do not hold, in general.

"But from the categorical perspective again we interpret a relational structure as a certain diagram of arrows “visualizing” the given relations between the objects which form the “nodes” of the diagram. It turns out that we can always “embed” such a diagram in an associated PATH-category having comparable behavior as the category associated to a reflexive, transitive relation, although being a little “bigger” concerning the morphism structure.

"We point out: The introduction of the associated category **PATH** allows to use and apply all the modeling principles and constructions provided by **CAT** in the corresponding situations."

## 2.2. Formal description

"For the technical definitions, again let R ⊂ X × X or (X,R) denote a (general) relation.
We associate to it the following category denoted by PATH(X,R) or just PATH for short, if no confusion can arise.

"The objects are the elements x ∈ X and arrows (morphisms) are defined by sequences (paths) of adjacent arrows. That means, there is always a morphism x -> y, if (x, y) ∈ R (or xRy), but if we have arrows (morphisms) x -> y and y -> z, then, in general we do not have a “direct arrow” x -> z, since the relation need not to be transitive. But what we can always do is forming a sequence (path) of consecutive arrows, like x -> y -> z in the previous case. This is then a morphism of a more general type between x and z.

"More generally, we can have (finite) sequences denoted for example by $x_0$ -> $x_1$ -> . . . -> $x_n$ which is a morphism in Mor($x_0$, $x_n$) in the new sense of our definition, but can also be interpreted as the composition of other morphisms which will be represented by adjacent parts of that whole path.

"In a category normally we need the identity arrow $id_x$ for each object x.
We can add this as a requirement if there is really a necessity from a theoretical viewpoint; in practice this may be irrelevant.

"In PATH it can be the case that there exists more than one path between two nodes a and b, therefore in PATH we can have for the sets of morphisms |Mor(a, b)| > 1, in general, in contrast to the category which is associated to a reflexive, transitive relation as considered before.
Based on these considerations we can see that **PATH** becomes a category.” (Pfalzgraf)

### 2.2.1. Formal description of PATH

"Let R ⊂ X × X denote a general relation. We associate with it the category denoted by PATH(X,R), PATH(X) or just PATH.
*Objects*: Elements x ∈ X.
*Arrows*, *Morphisms*: Sequences (paths) of consecutive arrows.

"This defines *composition* of arrows, in a natural way (concatenation of consecutive arrows) and this composition is associative. The identity arrow $id_x$, with respect to each object x ∈ X, will be assumed to exist (“tacitly”) by definition.

"There is a morphism x -> y, iff xRy.
In general, for arrows x -> y and y -> z, we do not have a “direct arrow” x -> z (the relation can be not transitive) - this causes no problem.

"We can always form a sequence (path) of *consecutive* arrows, like x -> y -> z.
This is a morphism of a more *general* type between x and z. More generally, we can have (finite) sequences, for example $x_0$ -> $x_1$ -> . . . -> $x_n$ (a path), this is a morphism in Mor($x_0$, $x_n$) in the new sense of our definition. It can also be interpreted as the composition of other morphisms being represented by adjacent parts of the long sequence. Thus, **PATH** is a category.” (Pfalzgraf)

# 3. Diamond of JOURNs

## 3.1. The journey map metaphor

Paths in diamonds are not exactly path in the sense of the category PATH but journeys. Thus, diamond paths are building diamond JOURNs. Hence, **JOURN** is not a category but a diamond.

*"To practice the complementarity of the movement is not as simple as it sounds. You have to have one eye in the driving mirror and the other eye directed to the front window and, surely, you have to mediate, i.e., to understand together, what you are perceiving: leaving and approaching at once. And the place you are thinking these two counter-movements which happens at once is neither the forward nor the backward direction of your journey. It’s your awareness of both. Both together at once and, at the same time, neither the one nor the other. It is your arena where you are playing the play of leaving and arriving."*

*"This complementarity of movements is just one part of the metaphor.*

*Because life is complex, it has to be composed by parts. Or it has to be de-composed into parts. We may drive from Dublin to Glasgow and then from Glasgow to London to realize our trip from Dublin to London. This, of course, is again something extremely simple to think and even to realize.*
*But again, there is a difference to discover which may change the way we are thinking for ever." (Kaehr, The Book of Diamonds, Intro)*
*http://rudys-diamond-strategies.blogspot.com*

**JOURN's catalogue of journeys**
There are structurally different kinds of journeys on offer.
1. PATH is a very special type of journey. It is an intra-contextural journey in a single contexture without structural environment. Hence, properly formalized as a category.
2. This situation might be distributed. Journeys in different but mediated contextures are possible. Still isolated and each thus intra-contextural.
3. A new kind appears with possible switches (permutation) and transjunctional splitting (bifurcation) simultaneously into paths of different contextures. Still without complementary environment in the sense of diamond theory.
3. Now, each contexture, even an isolated mono-contexture, might be involved into itself and its environment. This happens for diamonds, which are containing antidromically oriented path in categorical and saltatorial systems. Such journeys ar group-journeys with running into opposite directions.
3. Here, a new and risky journey is offered by the travel agency by inviting to use the bridging rules between complementary acceptional and rejectional domains of categories and saltatories of a diamonds. All that happens intra-contexturally, i.e. diamonds are defined as the complementarity of an elementary contexture.
4. Obviously, diamond journeys might be organized for advanced travellers into polycontextural constellations. Hence, there are transcontextural transitions between diamonds to risk. Interestingly, such journeys might be involved into metamorphic changes between acceptional and rejectional domains of different contextures of the polycontextural scenario.

**Further Metaphors**
As a metaphor, the idea of *colored* contextures, each containing a full PATH-system, involved in *interactions* between neighboring contextures, might inspire the understanding of journeys in pluri-labyrinths of JOURN.
Such journeys are not safely connected in the spirit of secured transitivity but are challenging by jumps, salti and bridging and transjunctional bifurcations and transcontectural transitions.
This metaphor of *colored* categories, logics, arithmetic and set theories gets a scientific implementation with real world systems containing incommensurable and incompatible but interacting domains, like for bio- and social systems.

***"Observation**: An arbitrary binary relation R on X induces a corresponding arrow diagram D, "visualizing" the given relations between objects by corresponding arrows. Vice versa, a given arrow diagram D induces (or defines) a corresponding binary relation R on the set of elements (nodes) of D in the obvious, natural way, i.e. a specific arrow x $\longrightarrow$ y in D defines xRy. In these situations there is always an associated Cat **PATH**."* (Pfalzgraf, ACCAT-TutorialSKRIPT, p. 32, 2004)

## 3.2. Formal description of JOURN

Let $R^{1.2} \subseteq (A_o^1, A_o^2)$ xx $(A_1^1, A_1^2)$, denote a general bi-relation. We associate with it the *diamond* denoted by JOURN((X,x), $R^{1.2}$ ), JOURN(X,x) or just JOURN.

*Bi-objects:* Bi-Elements (X,x ) ∈ ∈ **(X, x)**.

*Morphisms*: Sequences (paths) of consecutive arrows,
*Hetero-morphisms:*counter-sequences of antidromic arrows.

*Complementarity*: Category/Saltatory
**JOURN** is not a product of **PATH**, i.e. JOURN != PATH x PATH but a *complementary* (and not a dual!) interplay between PATH and co-PATH:

JOURN = compl(PATH, $\overline{\text{PATH}}$)

There is a *morphism* X -> Y, iff XRY∈ Cat .
There is a *hetero*-morphism x -> y, iff xry ∈ Salt.
There is a *diamond* if [Cat; Salt].

$$R^{1.2} \subseteq \left(A_o^1, A_o^2\right) \text{ xx } \left(A_1^1, A_1^2\right)$$
$$\left(\text{Rr}\right) \subseteq \subseteq \left(A_o^1, a_o^2\right) \text{ xx } \left(A_1^1, a_1^2\right)$$

### 3.2.1. Monocontextural diamond journeys

Diamond journeys might realize intransitivity by changing between categories and saltatories of a diamond. Also transitivity is established, taken in isolation, for both, the categorical composition of morphisms in a category and for the jump-operation (saltisition) of hetero-morphism in saltatories, journeys might follow nontraditional paths between categories and saltatories by exploiting the possibilities opened up by the *bridging* operations.

"Jumping operations are the main operations for hetero-morphisms. A new abstraction, additionally to composition and saltisition, is introduced for the *bridging* of categories and saltatories. Bridging has two faces: *bridge* and *bridging*."

**Collecting terms**
*Category*: composition based on matching conditions (coincidence)
*Saltatory*: saltisition based on jumping conditions
*Interactionality*: bridge, bridging, transversality, parallelity based on bridging conditions (difference).

**Possible chain of operators**
*composition* (o) produced by morphisms, matching condition, domain, codomain,
*saltisition* (||) produced by complementation (difference) of composition,
*bridge* (^) produced by composition and difference from category and saltatory,
*bridging* (•) produced by difference from bridge.

As a consequence, the *composition* (f o g) and the *saltisition* (k || l) are mixed to (l || k) o g).
Bridging vs. jumping shows clearly that not only *what* is achieved matters but *how* it is achieved, i.e. by bridging or by jumping. Each jump in a saltatory has an inverse morphism as a bridge in a category.

**Properties of bridging**

**Bridge and Bridging Conditions BC**

1. $\forall k, l, n \in \mathrm{HET}, \forall f, g, h \in \mathrm{MORPH}:$

*a.* **composition**

$g \circ f,\ g \circ h,$

$(h \circ g) \circ f,\ h \circ (g \circ f) \in \mathrm{MC},$

*b.* **saltisition**

$l \parallel k,\ n \parallel l,$

$n \parallel (l \parallel k),\ (n \parallel l) \parallel k \in \overline{\mathrm{MC}},$

*c.* **bridge**

$g \perp k,\ l \perp g,$

$(l \perp g) \perp k,\ l \perp (g \perp k)$ are in $\widehat{\mathrm{BC}}$.

*d.* **bridging**

$g \bullet k,\ l \bullet g,$

$(l \bullet g) \bullet k,\ l \bullet (g \bullet k)$ are in BC.

2. $(g \bullet k) \in \mathrm{BC}$ iff $\mathrm{dom}(k) = \mathrm{diff}(\mathrm{dom}(g))$,

$(l \bullet g) \in \mathrm{BC}$ iif $\mathrm{cod}(l) = \mathrm{diff}(\mathrm{cod}(g))$,

$(l \bullet g \bullet k) \in \mathrm{BC}$ iff $(g \bullet k),\ (l \bullet g) \in \mathrm{BC}$.

3. $(g \perp k) \in \widehat{\mathrm{BC}}$ iff $\mathrm{diff}(\mathrm{dom}(k)) = \mathrm{dom}(g)$,

$(l \perp g) \in \widehat{\mathrm{BC}}$ iff $\mathrm{diff}(\mathrm{cod}(l)) = \mathrm{cod}(g)$,

$(l \perp g \perp k) \in \widehat{\mathrm{BC}}$ iff $(g \perp k),\ (l \perp g) \in \widehat{\mathrm{BC}}$.

**Bridging**

*Associativity:*

If $k, g, l \in \text{BC}$, then $(k \bullet g) \bullet l = k \bullet (g \bullet l)$,

Bridging:

$$\text{bridging}_{(g,\, l,\, k)}:\ het(\omega_4, \alpha_4) \bullet hom(\alpha_2, \omega_2) \bullet het(\omega_8, \alpha_8) \longrightarrow het(\omega_9, \alpha_9).$$

***Bridge***

Associativity:

If $k, g, l \in \widehat{BC}$, then $(k \perp g) \perp l = k \perp (g \perp l)$,

*Bridge:*

$$\text{bridge}_{(g,\, l,\, k)}:\ \text{het}(\omega_4, \alpha_4) \perp \text{hom}(\alpha_2, \omega_2) \perp \text{het}(\omega_8, \alpha_8) \longrightarrow \text{het}(\omega_9, \alpha_9).$$

**Bridges vs. Bridging vs. Jumping**

$$(l \perp g \perp k) \cong (l \bullet g \bullet k) \cong (l \amalg k),$$
$$(l \perp g \bullet k) \cong (l \bullet g \perp k) \cong (l \amalg k),$$
$$(l \bullet g \perp k) \cong (l \perp g \bullet k) \cong (l \amalg k).$$
$$\text{diff}(\perp) = (\bullet),\ (\perp) = \text{diff}(\bullet).$$

## 3.2.2. Polycontextural journeys

Polycontextural category theory is enabling journeys between different contextures covering autonomous categories. This might happen in parallel or in cycles. Multiple parallel and cyclic journeys might running concurrently in the polycontextural matrix of disseminated categories.
For each category of a contexture an intra-contextural *path* might be realized and ruled by PATH.

An interplay between *local* and *global* thematization of the scenario, enabling distancing and zooming in, is describing intra- and poly-contexturality of JOURN.

$$\text{JOURN}^{(3)}{}_{(\text{cascade})}\left(\text{Cat}^{(3)}\right):$$
$$(\forall\ \forall\ \forall)\ A^{(3)}, B^{(3)}, C^{(3)}:$$
$$\frac{\left[\begin{array}{l} \text{Cat}^1 : A \pmb{\longrightarrow} B \circ B \longrightarrow C \Longrightarrow A \longrightarrow C \\ \text{Cat}^2 : A \longrightarrow B \circ B \pmb{\longrightarrow} C \Longrightarrow A \longrightarrow C \\ \text{Cat}^3 : A \longrightarrow B \circ B \longrightarrow C \Longrightarrow A \pmb{\longrightarrow} C \end{array}\right]}{\left(A \longrightarrow B \in \text{Cat}^1\right) \circ\circ\circ \left(B \longrightarrow C \in \text{Cat}^2\right) \Longrightarrow \left(A \longrightarrow C \in \text{Cat}^3\right).}$$

**Modeled in 3 - contextural predicate logic**

(C3) : (1 - contextural linear)

$\forall x \forall y ((Ez : K(x, y, z) \equiv (C(x) = D(y)))$

(C3a) : (3 - contextural parallel)

$\forall^{(3)} x \forall^{(3)} y :$

$((E^{(3)} z : K^{(3)} (x, y, z) \equiv (C^{(3)} (x) = D^{(3)} (y)))$

(C3b) : (3 - contextural cascade)

$\forall^{(3)} x \forall^{(3)} y :$

$((E^1 z : K^{(3)} (x, y, z) \equiv (C^1 (x) = D^2 (y))),$

$((E^2 z : K^{(3)} (x, y, z) \equiv (C^2 (x) = D^3 (y))).$

That is, for all 3-contextural parallel path-constellations (C3a) there is a cascade-journey (C3b) in (C3a).

$$\mathrm{JOURN}^{(3)}{}_{(cycle)} (\mathrm{Cat}^{(3)}) :$$

$$(\forall \forall \forall)\, A^{(3)}, B^{(3)}, C^{(3)} :$$

$$\frac{\left[\begin{array}{l} \mathrm{Cat}^1 : A \longrightarrow B \circ B \longrightarrow C \Longrightarrow A \longrightarrow C \\ \mathrm{Cat}^2 : A \longrightarrow B \circ B \longrightarrow C \Longrightarrow A \longrightarrow C \\ \mathrm{Cat}^3 : A \longrightarrow B \circ B \longrightarrow C \Longrightarrow A \longleftarrow C \end{array}\right]}{(A \longrightarrow B \in \mathrm{Cat}^1) \circ\circ\circ (B \longrightarrow C \in \mathrm{Cat}^2) \Longrightarrow (A \longleftarrow C \in \mathrm{Cat}^3).}$$

(C3c) : (3 - contextural cycle)

$\forall^{(3)} x \forall^{(3)} y :$

$((E^1 z : K^{(3)} (x, y, z) \equiv (C^1 (x) = D^2 (y))),$

$((E^2 z : K^{(3)} (x, y, z) \equiv (C^2 (x) = C^3 (y)))$

$((E^3 z : K^{(3)} (x, y, z) \equiv (C^{2.3} (x) = D^3 (y)))$

That is, for all 3-contextural parallel path-constellations (C3a) there is a (cascade) cyclic-journey (C3c) in (C3a).

### 3.2.3. Polycontextural diamond journeys

$$\text{JOURN}^{(3)}{}_{(\text{cycle})}\left(\text{Diam}^{(3)}\right):$$

$$U^{(3)} \frown \overline{U^{(3)}} = \emptyset$$

$$(\forall\forall\forall)\ A^{(3)}, B^{(3)}, C^{(3)} \in U^{(3)} \,\Big\|\, \forall^4\, d^{(3)} \in \overline{U^{(3)}}:$$

$$\frac{\begin{bmatrix} \text{Cat}^1: A \longrightarrow B \circ B \longrightarrow C \Longrightarrow A \longrightarrow C \\ \text{Cat}^2: A \longrightarrow B \circ B \longrightarrow C \Longrightarrow A \longrightarrow C \\ \text{Cat}^3: A \longrightarrow B \circ B \longrightarrow C \Longrightarrow A \longleftarrow C \end{bmatrix} \Bigg\| \begin{bmatrix} d_1 \longleftarrow d_2 : \text{Salt}^1 \\ d_1 \longleftarrow d_2 : \text{Salt}^2 \\ d_1 \longleftarrow d_2 : \text{Salt}^3 \end{bmatrix}}{\begin{bmatrix} (A \longrightarrow B) & - & - & - & - \\ & \circ & (B \longrightarrow C) & - & - \\ - & - & - & \circ & A \longleftarrow C \end{bmatrix} \Bigg\| \ d_1 \longleftarrow d_2}$$

**Short :**

$$\text{Cat}^{(3)}: \left(A \longrightarrow B \longrightarrow C \Longrightarrow A \longleftarrow C\right) \Big\| \left(d_1 \longleftarrow d_2\right): \text{Salt}^{(3)}.$$

(C3c) : (3 - contextural diamond cycle)

$$\forall^{(3)} x \,\forall^{(3)} y \in U^{(3)} \,\Big\|\, \forall^4\, v^{(3)} \in \overline{U^{(3)}}:$$

$$\left(\left(E^1 z : K^{(3)}(x, y, z) \equiv \left(C^1(x) = D^2(y)\right)\right)\right) \Big\| \left(d_1 \longleftarrow d_2\right)$$

$$\left(\left(E^2 z : K^{(3)}(x, y, z) \equiv \left(C^2(x) = C^3(y)\right)\right)\right) \Big\| \left(d_1 \longleftarrow d_2\right)$$

$$\left(\left(E^3 z : K^{(3)}(x, y, z) \equiv \left(C^{2.3}(x) = D^3(y)\right)\right)\right) \Big\| \left(d_1 \longrightarrow d_2\right)$$

with $\left(d_1 \longleftarrow d_2\right)^i,\ i = 1, 2, 3:$

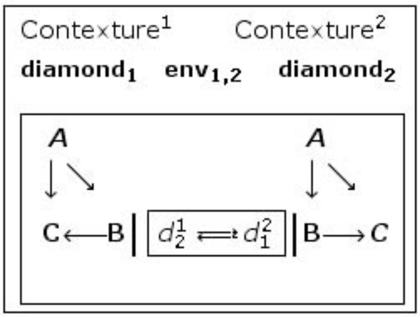
$$\Big\| E^4 v : \text{compl}\left(E^1 z : K^{(3)}(x, y, z)\right) \equiv$$

$$\text{diff}^i\left(C(x)\right) = \text{diff}^i\left(D(y)\right) \equiv$$

$$\left(d_1 \longleftarrow d_2\right)^i.$$

Is the diamond construction working for non-commutative compositions?
Diamond category theory was up to now based on the commutativity of composition of morphisms *f* and *g*, i.e. the coincidence relations for the composite *fg* was restricted to transitivity. Diamond structures are abstractions from compositions and are not depending on special properties of compositions. Hence, diamondization of relational concepts is well founded in the abstraction from composition in general.

### 3.2.4. Interplay between polycontextural diamond journeys




# 4. Diamond set theory

The concept of relation and its relational algebra is based on *set theory*. This wasn't always the case (Peirce). But today, set-theory based relation theory is well established.

Diamond relation theory, therefore, is based on diamond set theory. Diamond set theory is only in its very beginnings.

# 5. Architectonics of disseminated categories

## 5.1. Rhizomatics

The question behind the proposed constructions for non-commutative (diamond) category is unmasking a deeper structure of category theory not yet mentioned explicitly. That is: What is the general architectonics of categorical constructions? Or: On which architectonical decisions are categories based?
There are first answers to learn in the papers "*ConTeXtures*" and "*PolyLogics*".

### 5.1.1. Category theory

The leading metaphor of category theory can be found in the use of linerarily ordered compositions of arrows.
*"A category can be regarded as a directed graph with structure."*
They are symbolizing an information flow from arrow to arrow.
"*...the unifying idea is that of 'information flow'*.
Hence, "*Ordinary category theory uses 1-dimensional arrows - -->.*
*Higher-dimensional category theory uses higher-dimensional arrows.*"(Leinster, 2003)

### 5.1.2. n-Category theory

In contrast to category theory the leading metaphors of n-category is based in topology. With that, new fundamental or basic topologies or architectonics for categories are naturally motivated and constructed.
*"The natural geometry of these higher-dimensional arrows is what makes higher-dimensional category theory an inherently topological subject."* (Leinster, 2003)
Some further ideas are developed in the paper: "*Categories and Contextures*"
http://www.thinkartlab.com/pkl/lola/Categories-Contextures.pdf

### 5.1.3. Disseminatorics

Dissemination of categories over a *kenomic* matrix is not only topological but takes into account the mediation mechanism of proemiality instead of the quite vague matching conditions of category theory.

It could be stated as a theorem, that all possible relational constellations of combinatorial tree-structures can be represented as architectonics of disseminated categories.

Hence, all relational constellations, with commutative and diverse non-commutative properties, might serve as equal architectonics for disseminated, i.e. distributed and mediated, categories.
There is no prime structure.
A Birkhoff arithmetics of such skeletal structures might be considered.

Obviously, the game starts with the number 4. All other cases are structurally equivalent.
Especially for the number 3, there are no formal criteria to distinguish the skeletal architectures of a line from the architectures of a star, both are coinciding.

*"Only for the elementary case of m=3,* star *and* line *structures are coinciding. This simple structural coincidence may be the hidden reason of profound epistemological controversies in philosophy and sociology."* (cf. Kaehr, Contextures, 2005, Materialien 1978)

With the number 4, the skeletal difference of line- and star-structure is accessible. As the combinatorial table, below, for skeletal structures shows, an interesting coincidence of complexity and structures is given for the number 6. For complexity m>=7, the number of structures is increasing , structures(m)>m. E.g., m=10, structure(10) = 106.

The whole strategy could be called a *unification of relational and categorical structures* on a higher level of abstraction than relational and categorical notions. Instead of a lower level of "generalized morphisms" (Pfalzgraf) or the topological structures of n-categories.

**Again, modern strategies of defending paradigmatic fundamentalism**
As developed and argued at length at many places, polycontextural notions, theories and formalisms are not *fibered* theories, logics and semiotics, etc. (Pfalzgraf 1988), *"Polycontextural systems are topological fiberings of monocontextural systems" (*Toth 2009*)*, simply because fibering, in all its forms, is based on non-fibered category theory and predicate logic of a single universe.
Toth, Light, 2009:

It might sound more trendy to use fiber-terminology and techniques instead of clumsy many-sorted logical theories with identity (Goguen 1981), both are nevertheless fundamentally rooted in the uniqueness of a universe (of discourse, objects, elements, individuals, signs, marks, etc.) and there is no paradigmatic insight or strategy to abandon this kind of fundamentalism in favor of pluri-verses disseminated over kenomic matrixes in the sense of polycontextural and kenogrammatic endeavors and risks.
With more fun, read: SUSHI'S LOGICS, 2004.

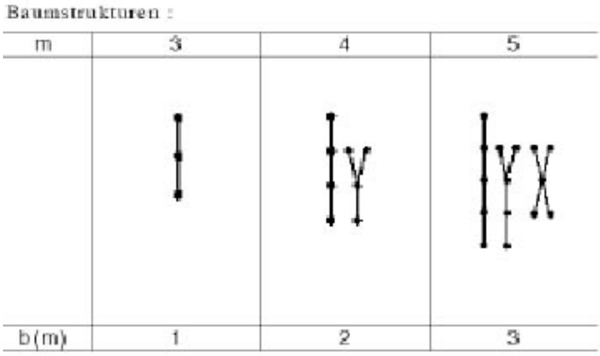
If there is something like a polycontextural category theory - with diamond structure or not - then there exists equally a polycontextural distribution of fibered and many-sorted theories as there exists, since at least 1978, a polycontextural dissemination of multivalued logics.

## 5.2. Dissemination between lines, stars, circles and flags

### 5.2.1. Skeletal structures

Arboreal patterns, linear and star-patterns (without root) are skeletal structures for the architectonics of dissemination.
A tabular and matrix approach to disseminated categories opens up a quite natural semantical interpretation of non-transitive mediated categorical systems.
**Table of skeletal structures**

Gerhard G. Thomas, "On Permatography", Proceeding of the 10th Winter School on Abstract Analysis (Srni 1982), Reyndiconti del Civcolo Matematico di Palermo.




Anzahlen :

| m | b(m) |
|---|---|
| 1 | 1 |
| 2 | 1 |
| 3 | 1 |
| 4 | 2 |
| 5 | 3 |
| 6 | 6 |
| 7 | 11 |
| 8 | 23 |
| 9 | 47 |
| 10 | 106 |
| 11 | 235 |
| 12 | 551 |
| 13 | 1301 |

A valuation of *skeletal* structures with the key distinctions of categories, like domain and codomain of morphisms, is enabling a distribution of categories over all possible combinatorial constellations. It is proposed that the skeletal structure of classical category theory is given by the skeletal *line*-structure with complexity m=3.

Skeletal structures are not rooted. For the purpose of modeling line- and star-structures for distributed categories the *directionality* of rooted graphs might be first omitted. That is, only one direction for *arrows* as interpreted graphs is chosen. For cyclic graphs, right and left orientations of graphs have to be involved. Questions of *knots* are not yet treated properly.

Hence, from the possibilities of the graph(3) = (•-•-•) and its arrow interpretation and its domain/codomain valuation, only the standard linear structure is considered as an architectonic base for category theory:

$$val_{arrow}(\bullet\text{-}\bullet\text{-}\bullet)=\begin{pmatrix}\bullet\longrightarrow\bullet\longrightarrow\bullet\\ \bullet\longrightarrow\bullet\longleftarrow\bullet\\ \bullet\longleftarrow\bullet\longrightarrow\bullet\\ \bullet\longleftarrow\bullet\longleftarrow\bullet\end{pmatrix} \text{ and } val_{dom/cod}\begin{pmatrix}\bullet\longrightarrow\bullet\longrightarrow\bullet\\ \bullet\longrightarrow\bullet\longleftarrow\bullet\\ \bullet\longleftarrow\bullet\longrightarrow\bullet\\ \bullet\longleftarrow\bullet\longleftarrow\bullet\end{pmatrix}.$$

That is, from $val_{dom/cod}(\bullet\longrightarrow\bullet\longrightarrow\bullet) = \begin{pmatrix}\square & - \\ \square & \square \\ - & \square\end{pmatrix}$ only the linear order

$$\begin{pmatrix}dom_1 & - \\ cod_1 & dom_2 \\ - & cod_2\end{pmatrix} \text{ with commutativity } \begin{pmatrix}dom_1 & - & dom_3 \\ cod_1 & dom_2 & - \\ - & cod_2 & cod_3\end{pmatrix} \text{ is considered.}$$

Both, left- and right-ordered linear structures, $(\bullet\longrightarrow\bullet\longrightarrow\bullet)$ and $(\bullet\longleftarrow\bullet\longleftarrow\bullet)$, are considered as isomorphic.
Also, left- and right-ordered star structures, $(\bullet\longrightarrow\bullet\longleftarrow\bullet)$ and $(\bullet\longleftarrow\bullet\longrightarrow\bullet)$, are considered as isomorphic.
Furthermore, linear and star structures coincide for m=3.

**General procedure**

$$val(graph) = distr[matrix]$$

$$architect: \{dom_i,\ cod_i,\ i \in N\} \longrightarrow [distribution-matrix]$$

**Example**

$$\text{val}\left(\longrightarrow\longrightarrow\right) = \begin{pmatrix} \square & - \\ \square & \square \\ - & \square \end{pmatrix}$$

$$\text{architect}\begin{pmatrix} \square & - \\ \square & \square \\ - & \square \end{pmatrix} = \begin{pmatrix} \text{dom}_1 & - \\ \text{cod}_1 & \text{dom}_2 \\ - & \text{cod}_2 \end{pmatrix}.$$

### 5.2.2. Line, star, cycles and flags

**Line$^{(3)}$ :**

$$\text{architect}_{\text{line}}^{(3)} = \begin{pmatrix} \text{dom}_1 & - \\ \text{cod}_1 & \text{dom}_2 \\ - & \text{cod}_2 \end{pmatrix}$$

$$\text{MC}_{\text{line}}^{(3)} = \{\text{cod}_1 \cong \text{dom}_2\}$$

$$\text{Hence, } \text{Cat}_{\text{line}}^{(3)} = \begin{pmatrix} \text{dom}_1 & - & \text{dom}_3 \\ \text{cod}_1 & \text{dom}_2 & - \\ - & \text{cod}_2 & \text{cod}_3 \end{pmatrix}$$

$$\text{Cat}_{\text{line}}^{(3)} : (\text{hom}) : \text{hom}(x, y) \circ \text{hom}(y, z) \Longrightarrow \text{hom}(x, z).$$

**Star$^{(3)}$ :**

$$\text{architect}_{\text{star}}^{(3)} = \begin{pmatrix} \text{dom}_1 & - \\ \text{cod}_1 & \text{cod}_2 \\ - & \text{dom}_2 \end{pmatrix}$$

$$\text{MC}_{\text{star}}^{(3)} = \{\text{cod}_1 \cong \text{cod}_2\}$$

$$\text{Cat}_{\text{star}}^{(3)} = \begin{pmatrix} \text{dom}_1 & - & \text{dom}_3 \\ \text{cod}_1 & \text{cod}_2 & - \\ - & \text{dom}_2 & \text{cod}_3 \end{pmatrix} \text{or} \begin{pmatrix} \text{dom}_1 & - & \text{cod}_3 \\ \text{cod}_1 & \text{cod}_2 & - \\ - & \text{dom}_2 & \text{dom}_3 \end{pmatrix}$$

$$\text{Cat}_{\text{star}}^{(3)} : (\text{hom}) : \text{hom}(x, y) \circ \text{hom}(x, z) \Longrightarrow \text{hom}(x, z) \vee \text{hom}(z, x).$$

$$\Longrightarrow \text{Cat}_{\text{line}}^{(3)} \cong \text{Cat}_{\text{star}}^{(3)}.$$

**Star$^{(4)}$ :**

$$\text{architect}_{\text{star}}^{(4)} = \begin{pmatrix} \text{dom}_1 & - & - \\ \text{cod}_1 & \text{dom}_2 & \text{dom}_3 \\ - & \text{cod}_2 & - \\ - & - & \text{cod}_3 \end{pmatrix}$$

$$\text{MC}_{\text{star}}^{(4)} = \left\{ \text{cod}_1 \cong \text{dom}_2 = \text{dom}_3 \right\}$$

$$\text{Hence, } \text{Cat}_{\text{star}}^{(4)} = \begin{pmatrix} \text{dom}_1 & - & - & - & \text{dom}_5 & \text{dom}_6 \\ \text{cod}_1 & \text{dom}_2 & \text{dom}_3 & - & - & - \\ - & \text{cod}_2 & - & \text{dom}_4 & \text{cod}_5 & - \\ - & - & \text{cod}_3 & \text{cod}_4 & - & \text{cod}_6 \end{pmatrix}$$

$$\text{Cat}_{\text{star}}^{(4)}(\text{hom}) = \begin{pmatrix} \text{hom}(x, y) \circ \text{hom}(y, z) \Longrightarrow \text{hom}(x, z) \\ \text{hom}(x, y) \circ \text{hom}(y, v) \Longrightarrow \text{hom}(x, v) \\ \text{hom}(y, z) \circ \text{hom}(z, v) \Longrightarrow \text{hom}(y, v) \end{pmatrix}$$

**Line$^{(4)}$:**

$$\text{architect}_{\text{line}}^{(4)} = \begin{pmatrix} \text{dom}_1 & - & - \\ \text{cod}_1 & \text{dom}_2 & - \\ - & \text{cod}_2 & \text{dom}_3 \\ - & - & \text{cod}_3 \end{pmatrix}$$

$$\text{MC}_{\text{line}}^{(4)} = \{\text{cod}_1 \cong \text{dom}_2, \text{cod}_2 \cong \text{dom}_3\}$$

$$\text{Cat}_{\text{line}}^{(4)} = \begin{pmatrix} \text{dom}_1 & - & - & \text{dom}_4 & \square & \text{dom}_6 \\ \text{cod}_1 & \text{dom}_2 & - & \square & \text{dom}_5 & \square \\ - & \text{cod}_2 & \text{dom}_3 & \text{cod}_4 & \square & \square \\ - & - & \text{cod}_3 & \square & \text{cod}_5 & \text{cod}_6 \end{pmatrix}$$

$$\text{Cat}_{\text{line}}^{(4)}(\text{hom}) = (\text{hom}(x, y) \circ \text{hom}(y, z) \circ \text{hom}(z, v) \Longrightarrow \text{hom}(x, v))$$

$$\begin{pmatrix} \text{hom}(x, y) \circ \text{hom}(y, z) \Longrightarrow \text{hom}(x, z) \\ \text{hom}(y, z) \circ \text{hom}(z, v) \Longrightarrow \text{hom}(y, v) \end{pmatrix} \Longrightarrow \text{hom}(x, v)$$

$$\Longrightarrow \text{Cat}_{\text{line}}^{(4)} \neq \text{Cat}_{\text{star}}^{(4)}$$

**Star – line$^{(5)}$ :**

$$\text{architect}_{\text{star-line}}^{(5)} = \begin{pmatrix} dom_1 & - & - & - \\ cod_1 & dom_2 & dom_3 & - \\ - & cod_2 & - & dom_4 \\ - & - & cod_3 & - \\ - & - & - & cod_4 \end{pmatrix}$$

$$MC_{\text{star-line}}^{(5)} = \left\{ cod_1 \cong dom_2 \cong dom_3 ,\ cod_2 \cong dom_4 \right\}$$

Null

$$Cat_{\text{star-line}}^{(5)} = \begin{pmatrix} dom_1 & - & - & - & dom_5 & - & - & dom_8 & - & dom_{10} \\ cod_1 & dom_2 & dom_3 & - & - & - & dom_7 & - & dom_9 & - \\ - & cod_2 & - & dom_4 & cod_5 & - & - & - & - & - \\ - & - & cod_3 & - & - & dom_6 & cod_7 & cod_8 & - & - \\ - & - & - & cod_4 & - & cod_6 & - & - & cod_9 & cod_{10} \end{pmatrix}$$

**Cycle$^{(4)}$**

$$\text{architect}_{\text{left-cycle}}^{(4)} = \begin{pmatrix} dom_1 & cod_2 & - & - \\ cod_1 & - & - & dom_4 \\ - & - & dom_3 & cod_4 \\ - & dom_2 & cod_3 & - \end{pmatrix}$$

$$\text{architect}_{\text{right-cycle}}^{(4)} = \begin{pmatrix} dom_1 & - & - & cod_4 \\ cod_1 & dom_2 & - & - \\ - & cod_2 & dom_3 & - \\ - & - & cod_3 & dom_4 \end{pmatrix}$$

$$\boxed{\text{architect}_{\text{left-cycle}}^{(4)} \neq \text{architect}_{\text{right-cycle}}^{(4)}}$$

**Flag$^{(4)}$**

$$architect^{(5)}_{right-cycle-flag} = \begin{pmatrix} dom_1 & - & - & cod_4 & - \\ cod_1 & dom_2 & - & - & - \\ - & cod_2 & dom_3 & - & - \\ - & - & cod_3 & dom_4 & dom_5 \\ - & - & - & - & cod_5 \end{pmatrix}$$

$$architect^{(5)}_{left-cycle-flag} = \begin{pmatrix} dom_1 & cod_2 & - & - & - \\ cod_1 & - & - & dom_4 & - \\ - & - & dom_3 & cod_4 & - \\ - & dom_2 & cod_3 & - & dom_5 \\ - & - & - & - & cod_5 \end{pmatrix}$$

**right – cycle – flag**

$$\begin{array}{ccccccccc} & & & & & & E & & \\ & & & & & & \uparrow & & \\ A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & A \end{array}$$

$$architect^{(5)}_{right-cycle-flag} \neq architect^{(5)}_{left-cycle-flag}$$

**Diamond – Star$^{(4)}$**

$$architect^{(4)}_{star} = \begin{pmatrix} dom_1 & - & - \\ cod_1 & dom_2 & dom_3 \\ - & cod_2 & - \\ - & - & cod_3 \end{pmatrix} \Longrightarrow \left( A \longrightarrow B \circ \begin{pmatrix} B \longrightarrow C \\ B \longrightarrow D \end{pmatrix} \right)$$

**Diamond – Star – Rule**

$$\frac{A \longrightarrow B \diamond \begin{pmatrix} B \longrightarrow C \\ B \longrightarrow D \end{pmatrix}}{A \longrightarrow B \circ \begin{pmatrix} B \longrightarrow C \\ B \longrightarrow D \end{pmatrix} \Bigg\| \begin{pmatrix} b_1^1 \longleftarrow b_2^1 \\ b_1^2 \longleftarrow b_2^2 \end{pmatrix}}$$

**Star$^{(4)}$ – rules**

$$\frac{A \longrightarrow B \circ \begin{pmatrix} B \longrightarrow C \\ B \longrightarrow D \end{pmatrix}}{A \longrightarrow C \;\Big|\; A \longrightarrow D \;\Big|\; C \longrightarrow D}$$

**Birkhoff arithmetics**

**Example**

$\text{architect}^{(5)}_{\text{star-line}} = \text{architect}^{(4)}_{\text{star}} \oplus \text{architect}^{(2)}_{\text{line}}$ :

$$\begin{pmatrix} \text{dom}_1 & - & - \\ \text{cod}_1 & \text{dom}_2 & \text{dom}_3 \\ - & \text{cod}_2 & - \\ - & - & \text{cod}_3 \end{pmatrix} \oplus \begin{pmatrix} \text{dom}_1 \\ \text{cod}_1 \end{pmatrix} = \begin{pmatrix} \text{dom}_1 & - & - & - \\ \text{cod}_1 & \text{dom}_2 & \text{dom}_3 & - \\ - & \text{cod}_2 & - & \text{dom}_4 \\ - & - & \text{cod}_3 & - \\ - & - & - & \text{cod}_4 \end{pmatrix}$$