The Book of Diamonds

Steps Towards a Diamond Category Theory

Experimental Sketch 0.1

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MOTIVATIONS for DIAMONDS at:
http://rudys-diamond-strategies.blogspot.com/2007/06/book-of-diamonds-intro.html
http://rudys-diamond-strategies.blogspot.com/2007/07/how-to-compose.html

Steps Towards a Diamond Category Theory

To accept the difference isn't easy; to enjoy it, a challenge.

1 Options of graphematic thematizations

1.1 Mono-contextural thematizations

Established as conflicts between dyads and monads.

1.2 Polycontextural thematizations

Introduced as a general theory of mediation.

1.2.1 Proemial thematizations

Realized as mediated triads of proposition / opposition and acceptance.

1.3 Diamond thematizations

Proposed as practicing the diamond, i.e., to diamondize.

An *example* of diamondizing object-oriented conceptualizations.

– Dyadic / monadic approach: MyClass = YourClass = Class

– Triadic Approach: Differences introduced as: [MyClass, YourClass, OurClass]

– Tetradic or Diamond approach: Transition from triadic to a tetradic appraoch with [MyClass, YourClass, OurClass, OthersClass]

1.3.1 Diamond class structure

$$\begin{array}{c} \overbrace{\text{OthersClass}} \\ \underbrace{\overbrace{\text{MyClass – YourClass}}} \\ \underbrace{\text{OurClass}} \end{array}$$

The harmonic My-Your-Our-Class conceptualization has to be augmented by a class which is incorporating the place for the other, the unknown, the difference to the harmonic system. That is, the NotOurClass is thematized positively as such as the class for others, called the *OthersClass*. Hence, the OthersClass can serve as the place where intruders, attacks, disturbance, etc. can be observed and defended. But also, it is the place where the new, inspiration, surprise and challenge can be localized and welcomed.

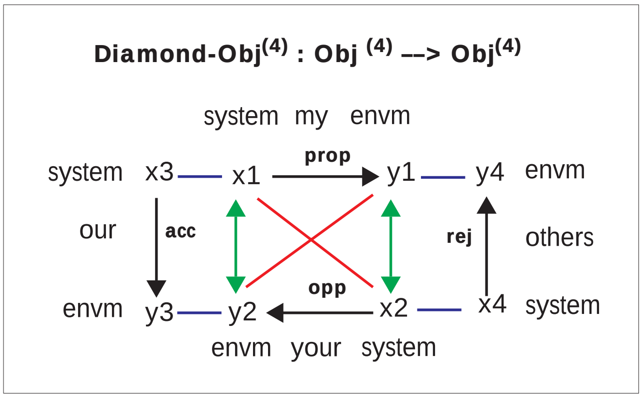
Again, this is a logical or conceptual place, depending in its structure entirely from the constellation in which it is placed as a whole. The OthersClass is representing the otherness to its own system. It is the otherness in respect of the structure of the system to which it is different. This difference is not abstract but related to the constellation in which it occurs. It has, thus, nothing to do with information processing, sending unfriendly or too friendly messages. Before any de-coding of a message can happen the logical correctness of the message in respect to the addressed system has to be realized.

In more metaphoric terms, it is the place where security actions are placed. While the OurClass place is responsible for the togetherness of the MyClass / YourClass interactions, i.e., mediation, the OthersClass is responsible for its segregation. Both, OurClass and OthersClass are second-order conceptualizations, hence, observing the complex core system "MyClass–YourClass". Internally, OurClass is focussed on what MyClass and YourClass have in common, OthersClass is focusing on the difference of both and its correct realization. In contrast to mediation it could be called *segregation*.

1.3.2 Diamond of system / environment structure

Some wordings to the diamond system / environment relationship.

What's my environment is your system,
What's your environment is my system,
What's both at once, my-systen and your-system, is our-system,
What's both at once, my-environment and your-environment, is our-environment,
What are our environments and our systems is the environment of our-system.
What's our-system is the environment of others-system.
What's neither my-system nor your-system is others-system.
What's neither my-environment nor your-environment is others-environment.



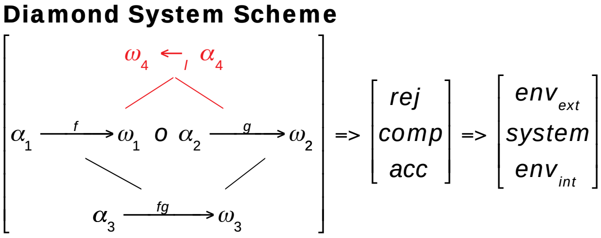
The diamond modeling of the otherness of the others is incorporating the otherness into its own system. An external modeling of the others would have to put them into a different additional contexture. With that, the otherness would be secondary to the system / environment complexion under consideration. The diamond modeling is accepting the otherness of others as a "first class object", and as belonging genuinely to the complexion as such.

Again, it seems, that the diamond modeling is a more radical departure from the usual modal logic and second-order cybernetic conceptualizations of interaction and reflection. The diamond is reflecting onto the same (our) and the different (others) of the reflectional system.

Internal vs. external environment

In another setting, without the "antropomorphic" metaphors, we are distinguishing between the system, its internal and its external environment. The external environment corresponds the rejectional part, the internal to the acceptional part of the diamond. Applied to the diamond scheme of diamondized morphisms we are getting directly the *diamond system scheme* out of the diamond-object model.

Diamond System Scheme



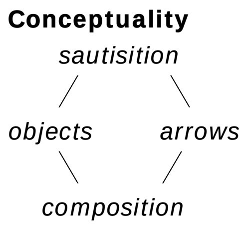
Thus, a diamond system is defined from its very beginning as being constituted by an internal and an external environment. ; the system *has* its own environment and is not simply inside or embedded into an environment.

reflectional / interactional

Further interpretations could involve the reflectional / interactional terminology of logics. The acceptional part fits together with the *interactional* and the rejectional part with the *reflectional* function of a system. Obviously, a composition is an interaction between the composed morphisms.

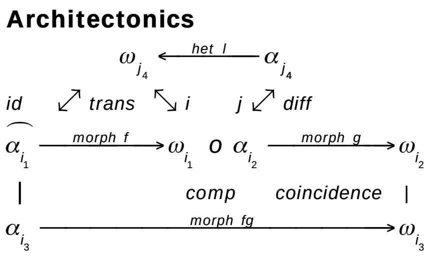
1.4 Prospect of Diamond Theory

Diamonds in this sketch are conceived as an interplay between categories and saltatories. Saltatories are the complementary concept of categories.



The conceptuality of diamond theory is introduced by an application of the *diamond strategies* to the basic concepts of category theory: *objects* and *morphisms* (arrows). Objects are understood in this setting as propositions, arrows as oppositions. Compositions appears as the both-at-once of objects and arrows, and sautisitions as the neither-nor of objects and arrows. Composition and sautisitions are complementary concepts.

Architectonics and terminology of diamond theory.



Terms

morph / het

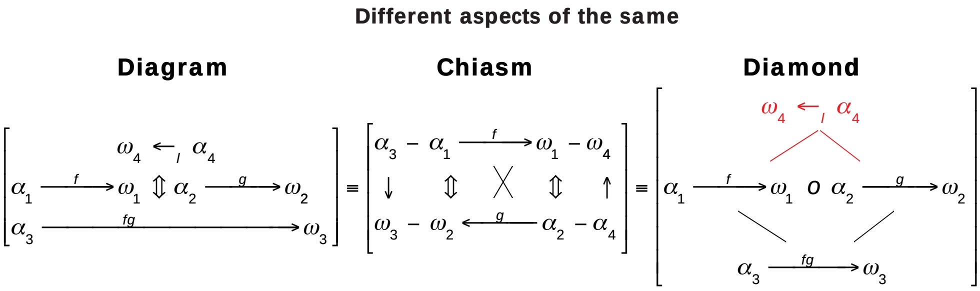
coinc / diff

id / div

o / ||

dual / compl

accept / reject



Diamond Theory DTh

Category : $\mathbf{A} = \left(Obj^{\mathbf{A}},\ hom,\ id,\ o\right)$

Saltatory : $\mathbf{a} = \left(Obj^{\mathbf{a}},\ het,\ id,\ ||\right)$

$DTh = \left(\left[\mathbf{A};\ \mathbf{a}\right],\ compl,\ diff,\ \bullet\right)$

Full Diamond

Category		Saltatory	
Cat	Cat^{op}	Salt	$Salt^{op}$

Categories are dealing with *composition* of morphisms and their laws. Saltatories are dealing with the jump-operation (*sautisitions*) of hetero-morphisms and their laws. Diamonds are dealing with the *interplay* of categories and saltatories. Their operation is interaction realized by the *bridging* operations.

Compositions as well as sautisitions (jump-operations) are ruled by *identity* and *associativity* laws. Complementarity between categories and saltatories, i.e., between acceptional and rejectional domains of diamonds, are ruled by *difference* operations.

Commutativity

$$\left(g \Diamond f\right) = \chi \left\langle \left(g \ o \ f\right) \| \ l \right\rangle \qquad \left(g \ o \ f\right) \| \ l:$$

with

$$\begin{pmatrix} cod\left(f\right) \triangleq dom\left(g\right) \\ \omega\left(f\right) \Updownarrow \alpha\left(g\right) \end{pmatrix} \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ & h \searrow & \downarrow g \\ & & C \end{array} \ \Bigg\| \ b_1 \xleftarrow{l} b_2$$

Diamond-composition is an interplay between category-composition and saltatory-composition.

Associativity Condition

If $f, g, h, k \in MC, \ l, m, n \in MC:$

$$then \begin{bmatrix} h \ o \left(\left(g \ o \ f\right) o \ k\right) = \left(\left(h \ o \ g\right) o \ f\right) o \ k \\ l \ \| \left(m \ \| \ n\right) = \left(l \ \| \ m\right) \| \ n \end{bmatrix}$$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow h & & \downarrow g \\ C & \xrightarrow{k} & D \\ \hline category & & \end{array} \ \Bigg\| \ \begin{array}{ccc} \overline{saltatory} & & \\ a & \xleftarrow{l} & b \\ & n \nwarrow & \uparrow m \\ & & c \end{array}$$

Distributivity

$$\left(k \ \| \ l\right) \bullet g = \left(g \bullet l\right) \| \left(g \bullet k\right)$$
$$\left(k \ \| \ l\right) \bullet g = \left(g \bullet l\right) o \left(g \bullet k\right)$$
$$\left(k \ \| \ l\right) \bullet g = \left(g \bullet l\right) \bullet \left(g \bullet k\right)$$

Between compositions and sautisitions laws of *distributivity* are established.

Transversality

$$transv_A : diff\left(A\right) \longrightarrow \left(B\right)$$
$$transv_B : A \longrightarrow diff\left(B\right)$$

Duality

$$\left[\mathbf{A}; \ \mathbf{a}\right] \in Diam : \left[\mathbf{A}; \ \mathbf{a}\right]^{op} = \left[\mathbf{A}^{op}; \ \mathbf{a}^{op}\right]$$

The duality of a diamond is realized by the duality of its category and the duality of its saltatory.

Complementarity

For $\forall X \in Diam:$

$X \in Acc \ iff \ compl\left(X\right) \in Rej,$

$compl\left(compl\left(X\right)\right) = X.$

Bridges vs. Bridging vs. Jumping

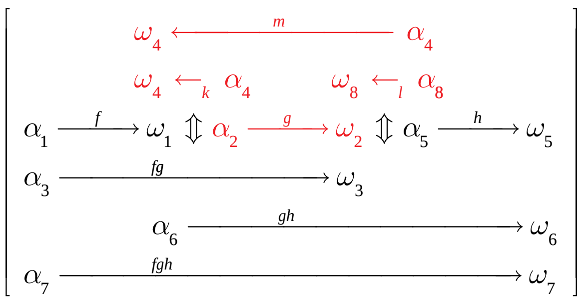
$$(l \perp g \perp k) \triangleq (l \bullet g \bullet k) \triangleq (l \parallel k),$$
$$(l \perp g \bullet k) \triangleq (l \bullet g \perp k) \triangleq (l \parallel k),$$
$$(l \bullet g \perp k) \triangleq (l \perp g \bullet k) \triangleq (l \parallel k).$$
$$diff(\perp) = (\bullet),\ (\perp) = diff(\bullet).$$

Jumping operations are the main operations for hetero-morphisms. A new abstraction, additionally to composition and saltisition, is introduced for the *bridging* of categories and saltatories. Bridging has two faces: *bridge* and *bridging*.

Double - face of Bridging

$$g \bullet (l \parallel k) \in BC,$$
$$(l \parallel k) \perp g \in \widehat{BC}:$$
$$g \bullet (l \parallel k) \triangleq (l \parallel k) \perp g$$

Bridging has two faces: *bridge* and *bridging*.

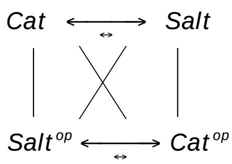


Proemiality

$$\forall i, j \in s(m):$$

$$\chi(Cat_i, Salt_j): \begin{bmatrix} [\mathbf{A};\ \mathbf{a}]_i \\ \updownarrow \ \times \ \updownarrow \\ [\mathbf{a};\ \mathbf{A}]_j \end{bmatrix} = \begin{bmatrix} [\mathbf{A} \to \mathbf{a}]_i \\ \updownarrow \ \times \ \updownarrow \\ [\mathbf{a} \to \mathbf{A}]_j \end{bmatrix}$$

Diamond Chiasm

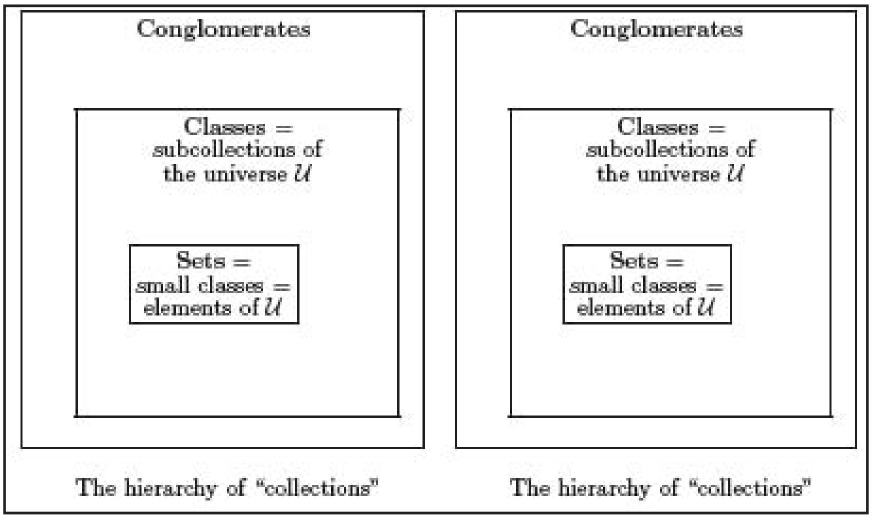


2 Diamonds and Contextures

It is said that category theory is a departure from set theory, other are more radical and insists that category theory has nothing to do with set theory at all.

From a foundational point of view, Herrlich makes it clear that a proper mathematical formalization of categories needs different sorts of *collections* of different generality. He distinguishes *sets*, *classes* and *conglomerates* as the collections appropriate to deal with categories.

A di-verse of collections



Collections of the universe U = [sets, classes, conglomerates].

The objects of category theory belong to these collections. Obviously, categorical objects are not simply sets but, e.g., categories of categories, hence surpassing all reasonable, i.e., contradiction-free notions of set theory. Hence, *"One universe as a foundation of category theory"*, (Mac Lane, 1969)

Diamond theory is in no way less general than category theory, but the objects of diamonds are not only collections of different degrees of abstractions, but are bi-objects from their very beginning. Bi-objects are complementary objects constructed as an interplay between acceptional and rejectional aspects of the diamond theory.

Hence the objects of diamonds are not simply belonging to the universe U of conglomerates with its classes and sets, but to the 2-verse (di-verse) as a complementarity of the universe of acceptional and the "universe" of rejectional objects.

Category theory happens in a universe, polycontexturality in a pluri-verse and diamond theory in a di-verse 2-U of complementarity.

Thus, 2-U = [collections || $\overline{\text{collections}}$].

Hence, 2-U = [(set || $\overline{\text{set}}$), (class || $\overline{\text{class}}$), (conglomerate || $\overline{\text{conglomerate}}$)].

A di-verse conception of collections opens up the possibility of *metamorphic* chiasms between their constituents [set, class, conglomerate]. This happens in a similar way like in polycontexturally disseminated categories. That is, a set in one contexture can be seen as a class in another contexture, etc. This happens on the base of the as-abstractions. In category theory as set is a set, a class is a class and a conglomerate is a conglomerate; and nothing else happens. The hierarchy is strict and well defined. The notions, set, class, conglomerate, are defined by is-abstractions.

This is different for polycontextural systems but also in diamond theory. For both, collections are still well defined and placed in their hierarchy. But because of the multitude of universes, interactions are possible between different kinds of collections. These interactions are strictly defined, too. They are ruled by the mechanism of chiastic metamorphosis.

Obviously, to describe the rules of sets, classes and conglomerates in di-verses we need some knowledge from diamond theory, which is based then just on such rules. That is, the whole idea of a di-verse is based on conceptions of diamond theory.

In diamond theory, conglomerates are not covering the situations of bi-objects. Bi-objects are polycontextural, thus they are members of disseminated conglomerates.

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Contexture(Conglomerate(Class(Set)))
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On the base of other conceptualizations of the diamond way of thematization, a transition from 2-verses to n-verses is not excluded. This should not be confused with the general multi-verses of polycontextural systems.

Diamond strategies for bi-objects

Bi-objects are strictly divided into a saltatorical and a categorical part. With the interplay and interactivity between categories and saltatories, ruled by the *bridging* conditions and operations, a new type of object emerges: bi-objects with mixed parts. Hence, diamonds are involved not simply in bi-objects but in bridges, too.

Bridges are composed by difference operation into a combination of categorical and saltatorical parts. In this sense, they are the both-at-once aspect of diamond bi-objects. A change of perspective in favor to the bridging operation as such, abstracting from its bi-objects, the neither-nor structure of bi-objects might be constructed.

Hence, we have to distinguish 4 aspects of diamonds: *categorical*, *saltatorical*, *interplay* (bridging as a mix) and *interactionality* (bridging as such).

2.1 Laws for sets

2.2 Laws for classes

2.3 Laws for conglomerates

2.4 Laws for universes

Universes are founded in uniqueness.

2.5 Laws for chiasms between universes

Metamorphic interchanges between universes, conglomerates, classes and sets.

3 Object-based Category Theory

Herrlich's definition of Category

3.1 DEFINITION

A **category** is a quadruple $\mathbf{A} = (\mathcal{O}, \text{hom}, id, \circ)$ consisting of

(1) a class $\mathcal{O}$, whose members are called **A-objects**,

(2) for each pair (A, B) of **A**-objects, a set $\text{hom}(A, B)$, whose members are called **A-morphisms from** A **to** B — [the statement "$f \in \text{hom}(A, B)$" is expressed more graphically[6] by using arrows; e.g., by statements such as "$f : A \to B$ is a morphism" or "$A \xrightarrow{f} B$ is a morphism"],

(3) for each **A**-object A, a morphism $A \xrightarrow{id_A} A$, called the **A-identity** on A,

(4) a **composition law** associating with each **A**-morphism $A \xrightarrow{f} B$ and each **A**-morphism $B \xrightarrow{g} C$ an **A**-morphism $A \xrightarrow{g \circ f} C$, called the **composite** of f and g,

subject to the following conditions:

(a) composition is associative; i.e., for morphisms $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, and $C \xrightarrow{h} D$, the equation $h \circ (g \circ f) = (h \circ g) \circ f$ holds,

(b) **A**-identities act as identities with respect to composition; i.e., for **A**-morphisms $A \xrightarrow{f} B$, we have $id_B \circ f = f$ and $f \circ id_A = f$,

(c) the sets $\text{hom}(A, B)$ are pairwise disjoint.

Comments

"If **A** = (O, *hom*, *id*, o) is a category, then

(1) The class O of **A**-objects is usually denoted by *Ob*(**A**).

(2) The class of all **A**-morphisms (denoted by *Mor*(**A**)) is defined to be the union of all the sets hom(A, B) in **A**.

(3) If f: A —> B is an **A**-morphism, we call A the **domain** of f [and denote it by *dom*(f)] and call B the **codomain** of f [and denote it by *cod*(f)].

Observe that condition (c) guarantees that each **A**-morphism has a *unique* domain and a *unique* codomain.

However, this condition is given for technical convenience only, because whenever all other conditions are satisfied, it is easy to "force" condition (c) by simply replacing each morphism f in hom(A, B) by a triple (A, f, B). For this reason, when verifying that an entity is a category, we will disregard condition (c).

(4) The composition, o, is a partial binary operation on the class Mor(**A**). For a pair (f, g) of morphisms, f o g is defined if and only if the domain of f and the codomain of g coincide." (Herrlich)

3.1 Description of the intuition

Descriptive definition of a diamond

Diamond Description

If $exch(\omega_1, \alpha_2)$, *and*

$$\begin{pmatrix} coinc(\alpha_1, \alpha_3), \\ coinc(\omega_2, \omega_3) \end{pmatrix},$$

then,

If $morph(\alpha_1, \omega_1)$ *o* $morph(\alpha_2, \omega_2)$ *then* $morph(\alpha_3, \omega_3)$,

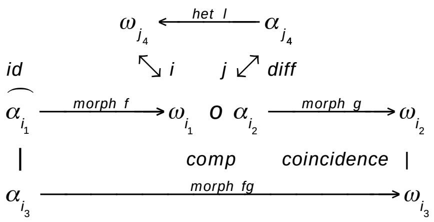
and if

$$\begin{pmatrix} diff(\alpha_2) = \alpha_4, \\ diff(\omega_1) = \omega_4 \end{pmatrix},$$

then

$$compl(morph(\alpha_3, \omega_3)) = het(\alpha_4, \omega_4).$$

Architectonics



Terms

morph / het
coinc / diff
id / div
o / ||
dual / compl
accept / reject

"Be-Wegung: weg von / Weg hin."

> "Jedoch, was heißt Weg, was heißt Unterwegssein?
> Der Weg: weg von / Weg hin (w / W).
> Das Wegen ermöglicht Weg, Ziel und Unterwegssein.
> Der Weg als methodos und das Wegen als Dekonstruktion des Weges der Methode.
> (Derridas Vorbehalte gegen die Methode als Weg mit Ziel.)"
> (Kaehr, DiamondStrategies-99)

John Baez' emphasis on the processuality of categorical concepts, like that an equality is in fact a process, is not thematizing the difference of open / closed worlds and their different concepts of iterability.

Janowskaya is introducing different concepts of iterability in her theory of different infinity constructions. But the change from one type of iterability to the next is not part of her theory. And *"finiteness of infinity"* is not yet thematized. A similar situation occurs with Warren McCulloch's speculations about finity and infinity.

3.2 Diamond composition

3.2.1 Morphisms with 2-objects

Morphisms as 2-objects consists of 2 pairs of distinctions:

1. domain (dom) and codomain (cod),
2. alpha and omega.

Thus, (g o f): cod(f) = dom(g) .simul. omega(f) exch alpha(g).

With diff(alpha(g))=dom(l) and diff(omega(f))= cod(l).

$$(g \diamond f) = \chi \langle g \ o \ f; l \rangle$$
$$iff$$
$$(g \ o \ f) \in MC = \begin{pmatrix} cod(f) \triangleq dom(g) \\ \omega(f) \Updownarrow \alpha(g) \end{pmatrix}$$

$$diff(\alpha(g)) = dom(l)$$
$$diff(\omega(f)) = cod(l)$$

3.2.2 Composition

Diamond Composite

$$\forall f, g : \begin{bmatrix} f : A \to B, \ g : B \to C \\ f : \alpha_1 \to \omega_1, \ g : \alpha_2 \to \omega_2 \end{bmatrix}$$

1. acceptional composite :

$$\forall f, g : f : A \to B, \ g : B \to C$$
$$cod(f) \equiv dom(g) \Rightarrow g \ o \ f : A \to C$$
$$dom(g \ o \ f) = dom(f)$$
$$cod(g \ o \ f) = cod(g).$$

2. rejectional composite :

$$\forall f, g : f : \alpha_1 \to \omega_1, \ g : \alpha_2 \to \omega_2$$
$$cod(f) \Updownarrow dom(g) \Rightarrow g \ o \ f : \alpha_3 \to \omega_3$$
$$compl(g \ o \ f) = compl\ (compl(g) \ o \ compl(f))$$
$$= compl(diff\ (cod(f)) \Updownarrow diff\ (dom(g)))$$
$$= compl((\overline{\omega_1}) \Updownarrow (\overline{\alpha_2})) = \alpha_4 \leftarrow \omega_4.$$

3. diamond composite :

$$Hence, \ \forall f, g : g \diamond f : [A \to C; \ \alpha_4 \leftarrow \omega_4]$$
$$\forall f, g : g \diamond f : [g \ o \ f; \ \overline{g \ o \ f}].$$

het / morph - Diamond Composition

1. het - composition

$\forall u, v : u : \omega_1 \leftarrow \alpha_1 , v : \omega_2 \leftarrow \alpha_2$

$\forall u, v : \left(u \parallel v\right) \in Comp :$

$cod\left(u\right) \cup dom\left(v\right) = \varnothing \Rightarrow u \parallel v : \omega_3 \leftarrow \alpha_3$

$dom\left(u \parallel v\right) = dom\left(u\right)$

$cod\left(u \parallel v\right) = cod\left(v\right).$

2. morph - composition

$\forall f, g, h : \left(h \circ g \circ f\right) \in Comp :$

$\forall f, g, h : f : A \rightarrow B, g : B \rightarrow C, h : C \rightarrow D$

$$\left.\begin{array}{l} cod\left(f\right) \equiv dom\left(g\right) \\ cod\left(g\right) = dom\left(h\right) \end{array}\right] \Rightarrow h \circ g \circ f : A \rightarrow D.$$

3. het / morph - interaction

$\left(u \parallel v\right) \in Comp$

iff

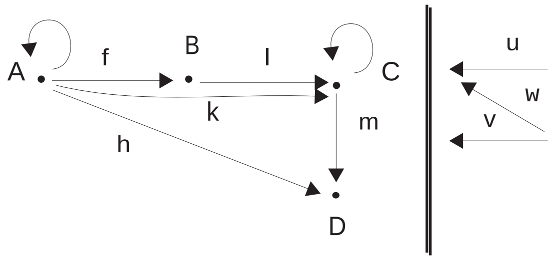
$\left(h \circ g \circ f\right) \in Comp.$

Diamond Composition

morph - composition	het - composition
$\forall f, g, h \in Morph :$ $f : A \rightarrow B, g : B \rightarrow C, h : C \rightarrow D$ $\forall f, g, h : \left(h \circ g \circ f\right) \in Comp_{Cat} :$ $cod\left(f\right) \equiv dom\left(g\right)$ $cod\left(g\right) = dom\left(h\right)$ $dom\left(h \circ g \circ f\right) = dom\left(f\right)$ $cod\left(h \circ g \circ f\right) = cod\left(h\right)$	$\forall u, v \in Het :$ $u : \omega_1 \leftarrow \alpha_1 , v : \omega_2 \leftarrow \alpha_2$ $\forall u, v : \left(u \parallel v\right) \in Comp_{Salt} :$ $cod\left(u\right) \cup dom\left(v\right) = \varnothing \Rightarrow u \parallel v : \omega_3 \leftarrow \alpha_3$ $dom\left(u \parallel v\right) = dom\left(u\right)$ $cod\left(u \parallel v\right) = cod\left(v\right)$
$\left(h \circ g \circ f\right) \in Comp_{Cat}$	$\left(u \parallel v\right) \in Comp_{Salt}$

$\forall f, g, h, \forall u, v : \left(h \lozenge g \lozenge f\right) = \left[\left(h \circ g \circ f\right) \mid \left(u \parallel v\right)\right] \in Comp_{Diam}$

3.3 Diamond Associativity



f o l = k, k o m = h
f o (l o m) = h = (f o l) o m)

rej(f o l) = rej(k) = u
rej(k o m) = rej(h) = v

rej(h) = rej(f o (l o m))
rej(h) = rej(f o rej(l o m))
rej(h) = u || rej(l o m))
rej(h) = u || v
f ∂ (l ∂ m) = [(f o(l o m);(u||v)]

rej(h) = rej((f o l) o m))
rej(h) = rej(rej(f o l) o m))
rej(h) = (rej(f o l) || v))
rej(h) = (u || v)
(f ∂ l) ∂ m) = [((f o l) o m);(u||v)]

Hence, (f ∂ l) ∂ m) = f ∂ (l ∂ m)

(u||v) = w, acc(w) = h, acc(u||v) = h = f o l o m
acc(u||v) = acc(u)||acc(v)
acc(u) = f o l, acc(v) = k o m
acc(acc(u)||acc(v)) = acc((f o l)||(k o m)) = (f o l) o (k o m)
acc(u||v) = (f o l) o ((f o l) o m)) = ((f o l) o (f o l)) o m)

acc(u||v) = (f o l) o m

4 Object-free categories

3.53 DEFINITION

An **object-free category** is a partial binary algebra $\mathbf{C} = (M, \circ)$, where the members of M are called **morphisms**, that satisfies the following conditions:

(1) *Matching Condition*: For morphisms f, g, and h, the following conditions are equivalent:

(a) $g \circ f$ and $h \circ g$ are defined,

(b) $h \circ (g \circ f)$ is defined, and

(c) $(h \circ g) \circ f$ is defined.

(2) *Associativity Condition*: If morphisms f, g, and h satisfy the matching conditions, then $h \circ (g \circ f) = (h \circ g) \circ f$.

(3) *Unit Existence Condition*: For every morphism f there exist units u_C and u_D of $(M, \circ)$ such that $u_C \circ f$ and $f \circ u_D$ are defined.

(4) *Smallness Condition*: For any pair of units (u_1, u_2) of $(M, \circ)$ the class $\hom(u_1, u_2) = \{f \in M \mid f \circ u_1 \text{ and } u_2 \circ f \text{ are defined}\}$ is a set.

3.54 PROPOSITION

If $\mathbf{A}$ *is a category, then*

(1) $(Mor(\mathbf{A}), \circ)$ *is an object-free category, and*

(2) an $\mathbf{A}$*-morphism is an* $\mathbf{A}$*-identity if and only if it is a unit of* $(Mor(\mathbf{A}), \circ)$.

Proof: $(Mor(\mathbf{A}), \circ)$ is clearly a partial binary algebra, where $f \circ g$ is defined if and only if the domain of f is the codomain of g. Thus each $\mathbf{A}$-identity is a unit. If $A \xrightarrow{u} B$ is a unit in $(Mor(\mathbf{A}), \circ)$, then $u = u \circ id_A = id_A$, where the first equality holds since id_A is a $\mathbf{A}$-identity and the second one holds since u is a unit. Thus (2) is established. From this, (1) is immediate. □

The "standard" and the "object-free" definitions of category are equivalent. For both definitions, the *sine qua non* is the *coincidence* of the co-domain and the domain of the morphisms to be composed. In the "object-free" definition the matching conditions for morphisms has to be matched. Any mismatch of the *"if and only if the domain of f is the codomain of g"* condition is destroying the category definitively.

Nevertheless, a purely "structural" or "operational" definition of category has to acknowledge that a target is not a source and a source is not a target. Their functionality are different, they are even opposites. Thus, to ask for a match or coincidence of a target (co-domain) and a source (domain) is abstracting from such fundamental differences. In favor of what? Let's say, of "objects", and their formal coincidence.

Diamonds are object-free. Their only objects are functional, i.e., categorial distinctions, alpha and omega of morphisms, and sameness and difference of distinctions. Nothing else. And this might emerge as the real departure from set-theory and object-orientedness. The idea of a *categorial* definition of categories goes back to my *Materialien 1973-75*, but at this time I didn't recognize the importance of the complementary construction of the "*jumpoids*".

4.1 Matching conditions

In this little sketch about a diamondization of the basic constructions of category theory some clarification of the basics of the diamond approach might be risked.

$$\begin{bmatrix} & \omega_4 \xleftarrow{l} \alpha_4 & \\ \alpha_1 \xrightarrow{f} & \omega_1 \Updownarrow \alpha_2 & \xrightarrow{g} \omega_2 \\ \alpha_3 & \xrightarrow{fg} & \omega_3 \end{bmatrix} \Rightarrow \begin{cases} CAT \ \ iff \ \ \omega_1 \triangleq \alpha_2 \\ DIAM \ \ iff \ \ \omega_1 \simeq \alpha_2 \end{cases}$$

A purely functional or operational thematization of the composition operation between morphisms has to make a difference between a strict, entity- or object-based, *coincidence*, and an operational based *difference* (similarity) between domain and codomain, target and source, of composed morphisms.

The concept of composition is fundamental for category theory, thus we have to start our diamond deconstruction with it. *"... category theory is based upon one primitive notion – that of* composition *of morphisms." D. E. Rydeheard*

Composition of morphisms as *coincidental*, and

Composition of morphisms as *differential*.

Or: Composition mode "sameness" and composition mode "difference".

Both modi, sameness and difference, together are defining a diamond category.

For diamonds, compositions of morphisms are realising both distinctions at once, the sameness and the difference of target and source, i.e., of composition.

For categories to work they have to realize the associativity conditions, which themselves are based on the matching conditions for the composition of morphisms.

> *"Associativity Condition:*
> If morphisms f, g, and h satisfy the matching conditions, then h o (g o f) = (h o g) o f."

The diamond approach is parallelizing the associativity conditions with the matching conditions. Instead of a succession of If-conditions, diamonds have to realize at once matching and associativity within their definition. This could be called an *in-sourcing* of the matching conditions into the definition of compositions. The main strategy to formalize diamonds should consider an interplay between matching conditions and assiociativity.

For morphisms f, g, h and k, associativity is realized only if associativity for acceptional and rejectional morphisms are realized at once. Hence, the interplay of acceptional and rejectional systems is choosing its matching conditions to realize associativity as a feature of diamonds. The strategy of formalizing diamonds should reverse the order of the categorical architecture. Not first morphisms, the matching conditions for compositions, then functors, then natural transformation, etc.

For classical categorical definitions, the matching conditions are *out-sourced* as *sine qua non* of compositions.

To follow, in analogy, step by step, the pre-given formalizations of categories to formalize diamonds is only a very first step towards a genuine diamond approach.

Matching conditions

$$\begin{aligned} &If\ \ cod(f) = dom(g) \\ &then: \\ &\qquad diff(cod(f)) \cong cod(l) \\ &\qquad diff(dom(g)) \cong dom(l) \\ &that\ \ is: \quad diff(g\ o\ f) \cong het(l) \end{aligned}$$

Domain and codomain of morphisms to compose have to match: cod(f)=dom(g).

Witin diamonds, morphisms have one "level" more, additional to a domain and codomain there is a diffential or rejectional level to each domain and codomain: diff(cod(f))=cod(l) and diff(dom(g))= dom(l), defining a hetero-morphism l.

Strictly, the domain and codomains distinctions of hetero-morphisms should be separated from their equivalents for morphisms because their objects are not belonging to the same universe of classes and sets.

Diamond Composition Definition

$$((k \lozenge h) \lozenge g) \lozenge f = k \lozenge (h \lozenge (g \lozenge f)):$$

$$\left\langle \begin{matrix} ((k\ o\ h)\ o\ g)\ o\ f \\ (l \parallel m) \parallel n \end{matrix} \right\rangle \triangleq \left\langle \begin{matrix} k\ o\ (h\ o\ (g\ o\ f)) \\ l \parallel (m \parallel n) \end{matrix} \right\rangle$$

Diamond Composition Derivations

$$1.\ (g \lozenge f) = \left\langle \begin{matrix} g\ o\ f \\ (f\ \bar{o}\ g) = l \end{matrix} \right.$$

$$2.\ (h \lozenge g) \lozenge f = h \lozenge (g \lozenge f):$$

$$\left\langle \begin{matrix} (h\ o\ g)\ o\ f = h\ o\ (g\ o\ f) \\ (m \parallel l) \end{matrix} \right.$$

Diamond Identity / Difference Definition

$$\lozenge_{ID} = \left\langle \begin{matrix} id_X : f\ o\ id_X = f = id_Y\ o\ f \\ diff_{fg} : (fg)\ o\ diff_{fg} = l = diff_{gf}\ o\ (gf) \end{matrix} \right.$$

Essential for the definition of the *category* is the composition operation and its associativity. Associativity enters the game with the composition of 3 morphisms.

In the same way, the definition of *diamonds* is ruled by the diamond composition and the necessity of 4 morphisms.

A composition in a *category* is defined by the coincidence of the codomain *cod* and the domain *dom* of the composed morphisms.

A composition in a *diamond* has always to reflect additionally the difference, i.e., the *complement* of the categorical composition operation. Thus, a diamond composition is producing a *composite* and a *complement* of the composed morphisms. The composite is the *acceptional*, and the complement the *rejectional* part of the diamond operation.

Morphisms with 2-objects

Morphisms as 2-objects consists of 2 pairs of distinctions:

1. domain (dom) and codomain (cod),
2. alpha and omega.

Thus, (g o f): cod(f) = dom(g) .simul. omega(f) ≠ alpha(g).

With diff(alpha(g))=dom(l) and diff(omega(f))= cod(l).

4.1.1 Identity and difference

$$(bi-object) \in \begin{bmatrix} id_{obj} \\ diff_{obj} \end{bmatrix}$$

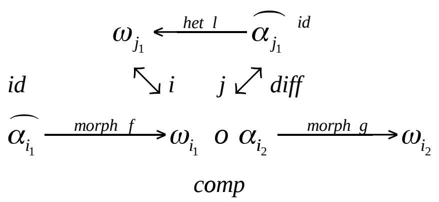
Identity is a mapping onto-itself as itself
Difference is a mapping onto-itself as other.

iff

$$\begin{bmatrix} id_{obj} \ o \ morph = morph \\ id_{obj} \ o \ het = het \\ diff_{obj} \ o \ morph = het \\ diff_{obj} \ o \ het = morph \end{bmatrix}$$

The formula "$diff_{obj}$ o morph = het" is an abbreviation for : "$diff_{obj}$ o ($morph_1$ o $morph_2$) = het".

General scheme



4.2 In-sourcing the matching conditions

Morphisms are representing mappings between objects, seen as domains and codomains of the mapping function.

Hetero-morphisms are representing the conditions of the possibility (Bedingungen der Möglichkeit) of the composition of morphisms. That is, the conditions, expressed by the matching conditions, are reflected at the place of the hetero-morphisms. Hetero-morphisms as reflections of the matching conditions of composition are therefore *second-order* concepts. Morphisms and their composition are *first-order* concepts, which have to match the matching conditions defined by the axiomatics of the categorical composition of morphisms. But these matching conditions are nor explicit in the composition of morphism but implicit, defined "outside" of the compositional system. Hence, in diamonds, the matching conditions of categories are explicit, and moved from the "outside" into the inside of the system.

In this sense, the rejectional system of hetero-morphisms is a reflectional system, reflecting the interactions of the compositions of the acceptional system. Hetero-morphisms are, thus, the "morphisms" of the matching conditions for morphisms.

Hetero-morphisms are "composed" by the jump operation, which is not interactional in the sense of the acceptional system.

Finiteness and Diamonds

The idea of *in-sourcing* the matching conditions into the definition of diamonds seems to be in correspondence with the two main postulates of *"Chinese Ontology"*, i.e., the permanent change of things and the finiteness or closeness of situations. That is, diamonds should be designed as structural explications of the *happenstance* of compositions and not as a succession of events (morphisms). More exactly, diamonds are contemplating the interplay of acceptional and rejectional thematizations. Thus, morphisms with their matching conditions and composability are in fact of secondary order for the understanding of diamonds.

The complementarity of construction and verification, which is happening at once and not in a temporal delay, is a consequence of the *finiteness* and *dynamics* postulate of polycontextural "ontology". This simultaneous interplay is based on the insight that a delayed verification (or testing in programming) would not necessarily verify the construction in question because, at least, the context will have changed in-between. Delayed verification is possible only in the very special case of frozen dynamics.

Hence, hetero-morphisms and rejectional systems and their interplay with acceptional systems in diamond constellations are a strict consequence of the structures of their temporality and ontology.

Matching conditions for Diamonds

$$a'.\ g\ o\ f,\ h\ o\ g,\ k\ o\ g\ are\ defined$$
$$iff \begin{bmatrix} g\ o\ f,\ h\ o\ g,\ k\ o\ g \\ l\ ,\ m\ ,\ n \end{bmatrix}$$

$$b'.\ h\ o\ ((g\ o\ f) o\ k)\ is\ defined$$
$$iff \begin{bmatrix} h\ o\ ((g\ o\ f) o\ k) \\ l \parallel (m \parallel n) \end{bmatrix}$$

$$b''.\ h\ o\ ((g\ o\ f) o\ k)\ is\ defined$$
$$iff$$
$$l \parallel (m \parallel n)\ is\ defined$$

$$c'.\ ((h\ o\ g)\ o\ f)\ o\ k\ is\ defined$$
$$iff$$
$$(l \parallel m) \parallel n\ is\ defined$$

$$\forall f,\ g,\ h,\ \forall u,\ v:$$
$$(h\ o\ g\ o\ f) \in MC_{acc}$$
$$iff$$
$$(u \parallel v) \in MC_{rej}$$

Composition of morphisms is defined, i.e., is an element of the matching conditions MC, if and only if their hetero-morphisms are defined. That is, composition is defined iff the interaction between morphisms and hetero-morphisms is realized. In the case of simple compositions and their single hetero-morphisms, the interplay between the different compositions (gof, hog, kog) and the hetero-morphisms (l, m, n) may not be very clear. Hence, the order given by the alphabetic order should be made explicit, say as n-tuples.

The interdependency of morphisms and hetero-morphisms is marked by the logical *"if and only if"* (iff), which is in this situation more or less a *metaphorical* use of logic because between acceptional and rejectional systems there is in fact no mono-contextural logical correlation.

4.2.1 How does the in-sourcing work?

A first answer was given in direct analogy to the associativity condition for morphisms.

2. Associativity Condition

$$a.\ If\ \ f, g, h \in MC,\ then\ h\ o\ ((g\ o\ f)\ o\ k) = ((h\ o\ g)\ o\ f)\ o\ k\ and$$
$$l, m, n \in MC \qquad\qquad l \parallel (m \parallel n) = (l \parallel m) \parallel n$$

For categories it seems to be clear that matching conditions (coincidences) are defining the composition of morphisms. For diamonds, with their double characterization, it seems to make sense that compositions are defining their matching conditions, too. Both, compositions and matching conditions, are in an interplay of mutual construction and verification. Hence, there is no circularity to state that matching conditions are defining composition and compositions are defining matching conditions because both are in a chiastic interplay, distributed over acceptional and rejectional abstraction-levels of the diamond.

$$a'.\ If\ \ f, g, h, k \in MC$$
$$l, m, n \in JC,$$
$$then \begin{bmatrix} h\ o\ ((g\ o\ f)\ o\ k) = ((h\ o\ g)\ o\ f)\ o\ k \\ l \parallel (m \parallel n) = (l \parallel m) \parallel n \end{bmatrix}$$

The matching conditions should be differentiated into matching conditions for morphisms (MC) and matching conditions for hetero-morphisms as jump-conditions (JC). Both are complementary to each other.

$$a''.\ h\ o\ ((g\ o\ f)\ o\ k) = ((h\ o\ g)\ o\ f)\ o\ k$$
$$iff$$
$$l \parallel (m \parallel n) = (l \parallel m) \parallel n$$

As a next step of in-sourcing the matching conditions into the diamond definition of associativity, the mutual implications of acceptional and rejectional compositions have to be implemented.

$$Diamond\ \ Associativity\ \ D-ASS$$
$$iff \begin{bmatrix} morph \in morph - ASS \\ het \in het - ASS \end{bmatrix}$$

In an other version, diamond associativity D-ASS is realized if and only if (iff) morphisms are elements of the class of morphism-associativity (morph-ASS) and at once hetero-morphisms are elements of the counter-class of hetero-morphism associativity (het-ASS). It would be to much of misleading wordings if this interplay would be modeled by a logical conjunction (and).

$$Diamond\ \ Associativity$$
$$iff\ \left[\chi(morph, het) \in ASS\right]$$

The interplay can be made explicit as a *chiasm* between morphisms and hetero-morphisms.

Diamond Associativity

$$morphisms\ k, g, h, f \in m - ASS$$
$$hetero - morphisms\ l, m, n \in h - ASS,$$
$$iff \begin{bmatrix} h\ o\ ((g\ o\ f)\ o\ k) = ((h\ o\ g)\ o\ f)\ o\ k \\ l \parallel (m \parallel n) = (l \parallel m) \parallel n \end{bmatrix}$$

To involve hetero-morphisms into associativity, diamonds need 4 morphisms on the acceptional level to produce 3 hetero-morphisms able to have the property of hetero-associativity.

Both together, in their interplay, written in brackets [-], are realizing diamond-associativity.

Operational definition of Diamond Category

A radical operational definition of Diamonds should get rid of any connections to set-theory. Thus, the matching condition based on sets has to be abandoned in favor to a functional matching, which is an exchange relation between alpha and omega of a morphism. Secondarily, the set-based matching can be re-introduced as a nivellation of the differences of alpha- and omega-functionality.

Diamond - Category DC

$$Category : \mathbf{A} = (Obj^{\mathbf{A}}, hom, id, o)$$
$$Jumpoid : \mathbf{a} = (Obj^{\mathbf{a}}, het, id, ||)$$
$$DC = ([\mathbf{A}; \mathbf{a}], diff, \bullet)$$

$$diff\,(comp\,(\alpha, \omega)_i) = het\,(\alpha, \omega)_j$$
$$diff\,(\alpha\ o\ \omega)_i = morph\,(\omega \leftarrow \alpha)_j$$

$$diff\,(comp\,(\alpha, \omega)_i) = het\,(\alpha, \omega)_j$$
$$diff\,(\alpha_1\ o\ \omega_2) = morph\,(\omega_3 \leftarrow \alpha_3)$$
$$If\ (\alpha_1\ o\ \omega_2)\ then \begin{pmatrix} diff\,(\alpha_1) = \alpha_3 \\ diff\,(\omega_2) = \omega_3 \end{pmatrix}$$

Diff is the difference of compl, i.e., the complementary composition function.

5 Properties of diamonds

5.1 Sub-Diamonds

4.1 DEFINITION

(1) A category $\mathbf{A}$ is said to be a **subcategory** of a category $\mathbf{B}$ provided that the following conditions are satisfied:

(a) $Ob(\mathbf{A}) \subseteq Ob(\mathbf{B})$,

(b) for each $A, A' \in Ob(\mathbf{A})$, $\hom_{\mathbf{A}}(A, A') \subseteq \hom_{\mathbf{B}}(A, A')$,

(c) for each $\mathbf{A}$-object A, the $\mathbf{B}$-identity on A is the $\mathbf{A}$-identity on A,

(d) the composition law in $\mathbf{A}$ is the restriction of the composition law in $\mathbf{B}$ to the morphisms of $\mathbf{A}$.

(2) $\mathbf{A}$ is called a **full subcategory** of $\mathbf{B}$ if, in addition to the above, for each $A, A' \in Ob(\mathbf{A})$, $\hom_{\mathbf{A}}(A, A') = \hom_{\mathbf{B}}(A, A')$.

A diamond A is said to be a sub-diamond of a diamond B provided that the following conditions are satisfied. Chiastic composition in diamonds are not excluding sub-set relations for sub-diamonds of diamonds. In a strict analogy to the category definitions of sub-categories, the definitions for sub-diamonds are introduced.

Sub - Diamonds

1. $\left[\mathbf{A};\ \mathbf{a}\right], \left[\mathbf{B};\ \mathbf{b}\right] \in Diam,\ then \left[\mathbf{A};\ \mathbf{a}\right] \subseteq \left[\mathbf{B};\ \mathbf{b}\right],\ if$

$(a)\ \ Ob\left(\mathbf{A};\ \mathbf{a}\right) \subseteq Ob\left(\mathbf{B};\ \mathbf{b}\right)$

$(b)\ \ \forall \mathbf{A},\ \mathbf{A}' \in Ob\left(\mathbf{A}\right), \forall \mathbf{a},\ \mathbf{a}' \in Ob\left(\mathbf{a}\right)$

$hom_{\mathbf{A}}\left(\mathbf{A},\ \mathbf{A}'\right) \subseteq hom_{B}\left(\mathbf{A},\ \mathbf{A}'\right)$

$het_{\mathbf{a}}\left(\mathbf{a}', \mathbf{a}\right) \subseteq het_{\mathbf{b}}\left(\mathbf{a}', \mathbf{a}\right)$

$(c)\ \ \forall \mathbf{A} - obj,\ id_{\mathbf{B}}\left(\mathbf{A}\right) = id_{\mathbf{A}}\left(\mathbf{A}\right)$

$\forall \mathbf{a} - obj,\ diff_{\mathbf{b}}\left(\mathbf{a}\right) = diff_{\mathbf{a}}\left(\mathbf{a}\right)$

$(d)\ \ comp\left(\mathbf{A};\ \mathbf{a}\right) = comp\left(\mathbf{B};\ \mathbf{b}\right) \setminus hom_{\mathbf{A}}\left(\mathbf{A},\ \mathbf{A}'\right)$

$comp\left(\mathbf{a}', \mathbf{a}\right) = comp\left(\mathbf{b}', \mathbf{b}\right) \setminus het_{\mathbf{a}}\left(\mathbf{a}', \mathbf{a}\right).$

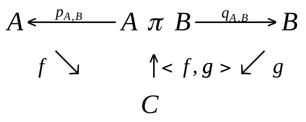
2. $\left[\mathbf{A};\ \mathbf{a}\right] \in full\ sub-diamond\ of\ \left[\mathbf{B};\ \mathbf{b}\right],\ if$

$\forall \mathbf{A},\ \mathbf{A}' \in Ob\left(\mathbf{A}\right),\ hom_{\mathbf{A}}\left(\mathbf{A},\ \mathbf{A}'\right) = hom_{B}\left(\mathbf{A},\ \mathbf{A}'\right)$

$\forall \mathbf{a},\ \mathbf{a}' \in Ob\left(\mathbf{a}\right),\quad het_{\mathbf{a}}\left(\mathbf{a}', \mathbf{a}\right) = het_{\mathbf{b}}\left(\mathbf{a}', \mathbf{a}\right).$

5.2 Diamond products

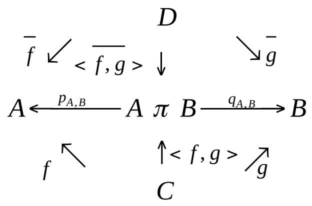
Categorical product



$$Logic^{1}:\begin{bmatrix}\forall C\\ \forall f,g\end{bmatrix}\exists!\langle f,g\rangle$$

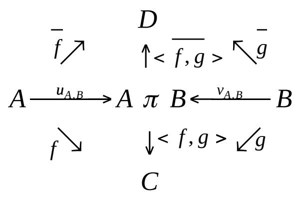
<f,g>: acceptance

Diamond product



$$Logic^{2}:\begin{bmatrix}\forall D\\ \forall \bar{f},\bar{g}\end{bmatrix}\exists!\langle \bar{f},\bar{g}\rangle$$

$<\overline{f,g}>$: rejectance



Diamond coproduct

5.3 Terminal and initial objects in diamonds

To each diamond, if there is a terminal object for its morphisms then there is a final object for its hetero-morphisms.

To each diamond, if there is a initial object for its morphisms then there is a final object for its hetero-morphisms.

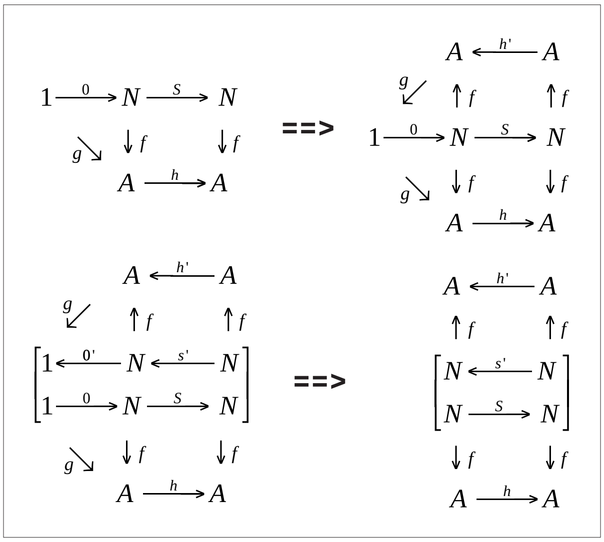
In diamond terms, rejectance has its own terminal and initial objects, like acceptance is having its own initial and terminal objects.

But both properties are distinct, there can be a final (terminal) object in a category, and another construction in a saltatory. Hence, the terms final and initial are not related to absolute concepts but to relative concepts depending on their context in a diamond.

Morphisms are ruled by equivalence; hetro-morphisms are ruled by bisimulation.

Equivalence belongs to the algebraic and constructive system (structure), bisimulation to the coalgebraic deconstructive system (process).

5.3.1 Towards a closure of initial objects



In an open world it wouldn't make much sense to run numbers forwards and backwards at once. But in a closed world, which is open to a multitude of other worlds, numbers are situated and distributed over many places and running together in all directions possible. Each step in a open/closed world goes together with its counter-step. There is no move without its counter-move.

If we respect the situation for closed/open worlds, then we can omit the special status of an *initial* object. That is, there is no zero as the ultimate beginning or origin of natural numbers in a diamond world. Everything begins everywhere. Thus, parallax structures of number series, where numbers are *ambivalent* and *antidromic*, are natural. It has to be shown, how such ambivalent and antidromic number systems are well founded in diamonds.

5.4 Functors between diamonds

5.4.1 Functors for categories

"Consider the category in which the objects are categories and the morphisms are mappings between categories. The morphisms in such a category are known as *functors*.

Given two categories, C and D, a functor F:C–>D maps each morphism of C onto a morphism of D, such that:

F preserves identities – i.e. if x is a C-identity, the F(x) is a D-identity

F preserves composition – i.e. F(f o g) = F(f) o F(g)." (Easterbrook)

In a diagram:

5.4.2 Functors for diamonds

In a similar wording, functors for diamonds are introduced.

Consider the diamond in which the objects are diamonds and the morphisms and hetero-morphisms are mappings between diamonds. The morphisms and hetero-morphisms in such a diamond, consisting of categories and saltatories, are introduced as *bi-functors*. Bi-functors are mappings in diamonds between categories and between saltatories.

Given two diamonds, C and D, a bi-functor 2-F:C–>D, [2-F: <C, c> –><D, d>], maps in the category each morphism of C onto a morphism of D, and in the saltatory each hetero-morphism of c onto a hetero-morphism of d such that:

Functor for categories:

F preserves identities – i.e. if x is a C-identity, the F(x) is a D-identity

F preserves composition – i.e. F(f o g) = F(f) o F(g).

Functor for saltatories:

F preserves differencies – i.e. if x is a c-difference, the F(x) is a d-difference

F preserves sautisition – i.e. F(f || g) = F(f) || F(g).

Functor for diamonds:

2-F preserves combination (comp, sautisition) – i.e. F(f o g)|| u) = F(f) o F(g) || F(u),

2-F preserves combination (bridging) – i.e. F(f • g)|| u) = F(f) • F(g) || F(u)

The bi-functor 2-F has to preserve the properties of categorical composition and saltatorical sautisition and also their combination (complementarity) including the mixing operations (bridge, bridging).

5.5 Natural Transformation and Diamonds

"Our slogan proclaimed: With each type of Mathematical object, consider also the morphisms. So, what is the morphism of functors; that is, a morphism from F to G where both F and G are functors F, G: ***C*** *–>* ***D*** *between categories* ***C*** *and* ***D****?"* MacLane, p. 390

"What sort of thing is the *"category of all categories"*?

"It turns out to be, not just a category, but a 2-category. That means that in addition to objects and morphisms, it has "2-morphisms", that is, morphisms between morphisms. To see how this goes, let's call the 2-category of all categories "Cat". Then the objects of Cat are categories, the morphisms of Cat are functors, and the 2-morphisms are natural transformations!" (Baez)
http://math.ucr.edu/home/baez/categories.html

Natural transformation

Definition 1.24 (Natural Transformation) Given the functors $F, G : \mathcal{A} \to \mathcal{B}$, a *natural transformation* $\alpha : F \Rightarrow G$ consists of a collection of arrows $\langle \alpha_C : FC \to GC \rangle_{C \in \mathcal{A}}$ in $\mathcal{B}$, such that for any arrow $h : X \to Y$ in $\mathrm{arr}(\mathcal{A})$ the diagram

$$\begin{array}{ccc} FX & \xrightarrow{\alpha_X} & GX \\ \downarrow{\scriptstyle Fh} & & \downarrow{\scriptstyle Gh} \\ FY & \xrightarrow[\alpha_Y]{} & GY \end{array}$$

commutes, *i.e.*, $\alpha_Y \circ Fh = Gh \circ \alpha_X$. The diagram above is the *naturality square* associated with h.

Notation 1.25 A natural transformation $\alpha : F \Rightarrow G$ for functors $F, G : \mathcal{A} \to \mathcal{B}$ is a family of arrows in $\mathcal{B}$ indexed by objects in $\mathcal{A}$. Each arrow of such a family is a *component* of the natural transformation, and we use subscripts to denote them, *e.g.*, $\alpha_C : FC \to GC$. We use angle brackets to denote the family of arrows as in the definition $\langle \alpha_C : FC \to GC \rangle_{C \in \mathcal{A}}$. Naturality squares are often depicted with the arrow which they are associated with on one side, recall that the arrow and the naturality square may be from different categories.

BRICS Lecture Notes, LS-02-catnote, 2002

Hence, the new slogan, additionally to Mac Lane's, could be:

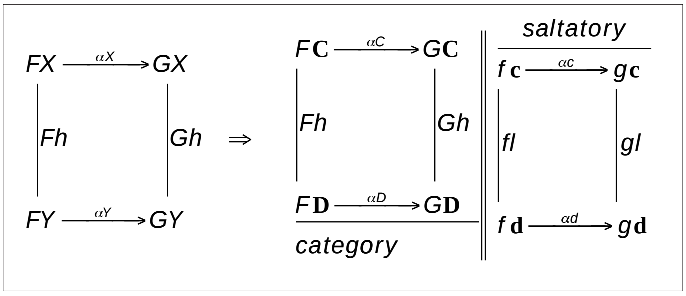
With each type of mathematical morphism, with its mathematical objects, consider also the diamonds *of the combinations of morphisms.*

So, what is the Diamond of Natural Transformations?

So, what is the diamond-morphism of functors; that is, a morphism from F to G where both F and G are functors F, G: ***C*** *–>* ***D*** *between diamonds* ***C*** *and* ***D****?*

First we have to consider diamonds as interplaying combinations of categories and saltatories. Hence, diamonds are bi-combinations of categories and saltatories, and categorical functors (F, G) appears as bi-functors between diamonds.

Transition from categorical to diamond "natural transformation"



Functors	Diamonds
F => <F, f>	**C** => **(C, c)**
G => <G, g>	**D** => **(D, d)**

$$\langle \alpha \rangle_{(D)} \; o \; \langle F \rangle \langle h \rangle = \langle G \rangle \langle h \rangle \; o \; \langle \alpha \rangle_{(C)} \; \Big\| \; \langle \beta \rangle_{(d)} \; o \; \langle f \rangle \langle l \rangle = \langle g \rangle \langle l \rangle \; o \; \langle \beta \rangle_{(c)}$$

$$\langle \alpha, \beta \rangle_{(D, d)} \; o \; \langle F, f \rangle \langle h, l \rangle = \langle G, g \rangle \langle h, l \rangle \; o \; \langle \alpha, \beta \rangle_{(C, c)}$$

Distinctions in respect of alpha and gamma, and h and l, have to be considered.

What has to be added is a concept of functors between categorical and saltatorical functors, i.e., a diamond-functor of the interplay of categories and saltatories.

Hence, the new functor (of functors) is $: h —> l

One candidate is complementarity.

6 Aspects of diamonds

Diamonds are produced by the interplay of acceptional and rejectional parts. Acceptional parts correspond to categories, and rejectional parts are corresponding to saltatories. Another thematization considers that diamonds consists of 3 parts: the *core* systems, the *acceptional* and the *rejectional* parts.

Core systems, as compositions of morphisms are in this respect the basic systems. They might have the property of transitivity (commutativity) and associativity. But these properties are result of a specific interpretation of the linear composition structure of the core system. Other properties, instead of transitivity and associativity, are possible for linear compositions. This may depend on the definition of the identity function ID.

Acceptional systems, therefore, have an own status as specific properties of core systems. Their properties, combined with the core system, are studied by *category theory*.

Rejectional systems, hence, also acceptional systems haven't been recognized until now, they have an equal legitimacy like the acceptional systems. Thus, they represent another set of properties of core systems. The properties of rejectional systems, combined with their core systems, are studied by *saltatory theory*.

Complementarity of acceptional and rejectional systems are a topic to be studied.

Diamond theory is studying the properties of the complementarity of acceptional and rejectional systems as an interplay of category and saltatory theory.

These are the first-order properties of diamonds. Their "data" are morphisms and hetero-morphisms, their "structure" composition and identity. Additional to the category theoretic distinction of *Data*, *Structure*, *Property* (DSP), diamond theory is considering the "meta-property" of the *Interplay* of saltatories and categories, hence, the diamond system is characterized by diamondized DSPI.

Second-order properties of diamonds are accessible by diamondization. The diamondization of diamonds is discovering new properties of diamonds.

Localization of diamonds in the contextural and kenomic grid with its tectonic of proto-, deutero- and trito-structure has to be considered. The localization of diamonds in the tabular position-system is ruled by its system of "*place-designators*".

6.1 Data, Structure, Property (DSP) for Categories

1.1.1 Categories I: graphs with structure

Definition 1 *A category is given by*

i) DATA*: a diagram* $C_1 \overset{s}{\underset{t}{\rightrightarrows}} C_0$ *in* Set

ii) STRUCTURE*: composition and identities*

iii) PROPERTIES*: unit and associativity axioms.*

The data $C_1 \overset{s}{\underset{t}{\rightrightarrows}} C_0$ is also known by the (over-used) term "". We can interpret it as a set C_1 of arrows with source and target in C_0 given by s, t.

6.2 Data, Structure, Property, Interactionality (DSPI) for Diamonds

DSPI-List

i) *Data*: 2-diagram C1–$_{s,t}$—>Co / Co<–$_{diff}$–C1 in 2-Set
ii) *Structure*: composition, identities + jump, difference
iii) *Properties*: unit, associativity + diversity, jump law
iv) *Interplay*: chiasm between category and saltatory.
(v) *Interactions*: diamonds with diamonds, iterative / accretive
vi) *Localisation:* kenomic grid, place-designator

DSPI-Explications

i) *Data*: 2-diagram C1–$_{s,t}$—>Co / Co<–$_{diff}$–C1 in 2-Set

Diamond - Data

$$C_0 \xrightarrow[target]{source} C_1 : Set_{Salt}$$
$$\Updownarrow$$
$$C_1 \xrightarrow[target]{source} C_0 \ : Set_{Cat}$$

Objects in diamonds are involved into 2 operations: coincidence and difference.

Coincidence is enabling composition and therefore, commutativity.

Differences are enabling hetero-morphisms and therefore jumpoids (jump commutativity).

Each object is involved in a difference and double identity relation.

ii) *Structure*: commutative composition, identities + complement, differences

commutative composition – complementation

$$h = (g \ o \ f) \,\|\, k :$$
$$\begin{bmatrix} A \xrightarrow{f} B \xrightarrow{g} C \\ b_1 \xleftarrow{k} b_2 \end{bmatrix}$$

$$\begin{matrix} A \xrightarrow{f} B & \\ h \searrow \quad \downarrow g & \Bigg\| \ b_1 \xleftarrow{k} b_2 \\ C & \end{matrix}$$

Communicative composition of morphisms in categories is based on a *binary* operation

hom(X, Y) x hom(Y, Z) —> hom(X, Z).

Composition in diamonds is based on a "*ternary*" operation "composed" by composition and complementation of composition:

hom(X, Y) [x $\overline{x}$] hom(Y, Z) —> hom(X, Z) | | $\overline{\text{hom}}$(X, Z).

identity – difference

Each object of a diamond is involved in a difference and double identity relation. Hence, diamond objects as bi-objects are polarities, i.e., their inner structure is that of a complementary polarity.

Identity rule

$$\frac{idX \xrightarrow{morph} Y \qquad X \xrightarrow{morph} Yid}{X \xrightarrow{morph} Y}$$

Difference rule

$$\frac{diffX \xrightarrow{morph} Y \qquad X \xrightarrow{morph} Ydiff}{X \xleftarrow{het} Y}$$

The difference operation separates the polarity of the bi-object into its acceptional and its rejectional parts (aspects).

Diamond objects are not only involved into *right* and *left* identity but in *transversal* difference.

iii) *Properties*: unit, associativity + diversity, jump law

Diamond

$$\left.\begin{array}{c} A \xrightarrow{f} B \\ \downarrow h \qquad \downarrow g \\ C \xrightarrow{k} D \\ \hline category \end{array}\right\| \begin{array}{c} saltatory \\ \hline a \xleftarrow{l} b \\ \nwarrow n \quad \uparrow m \\ c \end{array}$$

Strict simultaneity of categories and saltatories.

Meta-properties

iv) *Interplay*: chiasm between category and saltatory.

Interactivity

$$\begin{bmatrix} (k \,\|\, l) \ o \ g \\ g \ o \ (k \,\|\, l) \end{bmatrix} = \begin{bmatrix} (g \,\|\, l) \ o \ (g \,\|\, k) \\ (g \ o \ l) \,\|\, (k \ o \ g) \end{bmatrix}$$

Distributivity

$$(k \,\|\, l) \bullet g = (g \bullet l) \,\|\, (g \bullet k)$$
$$(k \,\|\, l) \bullet g = (g \bullet l) \ o \ (g \bullet k)$$
$$(k \,\|\, l) \bullet g = (g \bullet l) \bullet (g \bullet k)$$

Bridges and bridging operators are ruling the interplay between categories and saltatories as a mix of both.

v) *Interactions*: diamonds with diamonds

Iterative interactions

accretive interactions

metamorphic interactions

vi) *Localisation:* kenomic grid (proto-, deutero-, trito-structure, place-designator

Objects – Morphisms – Interactions//Structures – Properties

For diamonds, the categorical architectonics of DSP has to be reversed to IPSD:
First are *interactions* between diamonds, iterative and accretive compositions,
Second are interplays between categories and saltatories, with bridge and bridging,
third, morphisms/hetero-morphisms happens between objects.
Interactions have structures and properties.

6.3 Diamondization of diamonds

Like the possibility of categorization of categories there is a similar strategy for diamonds: *the diamondization of diamonds.* Categorizations and diamondizations are activities producing the conceptual fields for category and diamond theory. Diamond strategies are opening up the worlds of diamond theories. As a self-application of the diamond questions, the diamond of the diamond can be questioned. Diamond are introduced as the quintuple of proposition, opposition, acceptionality, rejectionality and positionality, D=[prop, opp, acc, rej; pos].

The complementarity of *acceptional* and *rejectional* properties of a diamond can themselves be part of a new diamondization.

What is both together, acceptional and rejectional systems? As an answer, *mediating* systems can be considered as belonging at once to acceptional as well to rejectional systems.

What is neither acceptional nor rejectional? An answer may be the *positionality* of the diamond. Positionality of a diamond is neither acceptional nor rejectional but still belongs to the definition of a diamond.

Hence, diamond of diamonds or second-order diamonds:

DD=[Acc, Rej, Med, Pos].

Thus,

[Acc, Rej]-*opposition* can be studied on a second-level as a complementarity per se,

[Acc, Rej]-*both-and* can be studied as the core systems per se (Med),

[Acc, Rej]-*neither-nor* can be studied as the mechanisms of positioning (Pos), esp. by the place-designator.

What are the specific formal laws of the diamond of diamonds?

Between the first-order opposition of acceptional and rejectional systems of diamonds there is a complementarity, which can be studied as such on a second-level of diamondization. What are the specific features of this complementarity? Like category theory has its *duality* as a meta-theorem, second-order diamond theory has its *complementarity* theorem.

Hence, it is reasonable to study core systems per se, without their involvement into the complementarity of acceptional and rejectional systems. What could it be? Composition without commutativity and associativity? The axioms of identity and associativity are specific for categories. But, on a second-order level, they may be changed, weakened or augmented in their strength.

The study of the positionality per se of diamonds might be covered by the study of the functioning of the place-designator as an answer to the question of the positionality of the position of a diamond. Without doubt, positionality and its operators, like the *"place-designator"* and others, in connection to the kenomic grid, can be studied as a topic per se.

The first-order positionality of diamonds has become itself a topic of second-order diamonds, the neither-nor of acceptance and rejectance. Hence, because also second-order diamonds are positioned, a new kind of localization enters the game: the localization of second-order diamonds into the tectonics of kenomic systems, with their proto-, deutero- and trito-kenomic levels.

All together is defining a second-order diamond theory.

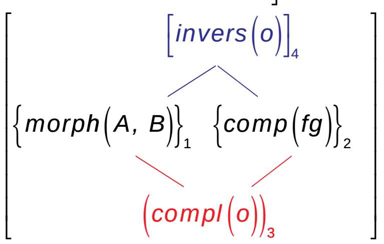
6.4 Conceptual graphs of higher-order diamondizations

6.4.1 Intrinsic higher-order Diamonds

A kind of a higher-order diamondization is introduced by the basic terms of diamondization: *morphism, composition, duality, complementarity, inversion.*

Diamond of

[morph, comp, compl, invers]

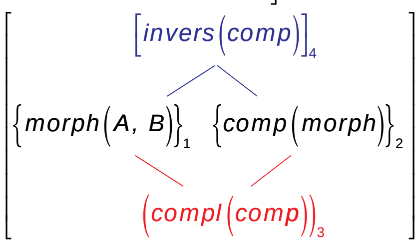


morph(A, B) = *morphisms* between A and B,
comp(fg) = *composition* of morphisms f, g,
compl(o) = *complement* of composition (f o g)
invers(o) = *morphogram* of compositor (o).

A different notation is focusing more on the operators of diamonds (morph, comp, compl, invers) instead of the operands (A, B, f, g) of the previous graph.

Diamond of

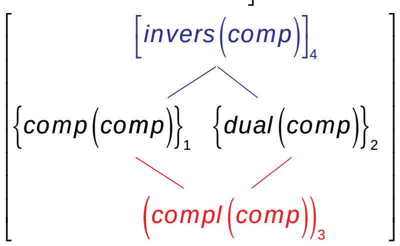
[morph, comp, compl, invers]



invers(comp) could also be seen as
invers(compl(comp)), i.e.,
invers(comp)$=_{mg}$invers(compl(comp))

Diamond of

[comp, dual, compl, invers]



One more abstraction is achieved with the transition to the diamond of the main operations over compositions of morphisms: *compositionality, duality, complementarity and subversionality.*

– *comp(comp)* is realizing *categories as* compositions of morphisms,

– *dual(comp)* is realizing the *duality* of a category.

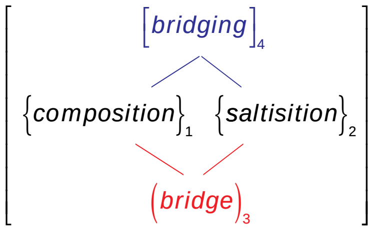
– *compl(comp)* is realizing *saltatories*, and

– *invers(comp)* is introducing the *morphogrammatics* of categories and saltatories.

Operational diamond

Diamond of

[*comp, salt, bridge, bridging*]



A further diamond is introduced on the base of its primary operation: composition, saltisition, bridge and bridging.

Bridges are combination of categorical and saltatorical parts based on the *difference* operation. In this sense, they are the *both-at-once* aspect of diamond bi-objects functioning as a bridge, i.e., an *interplay* between composition and saltisition. A change of perspective in favor to the bridging operation as such, abstracting from its bi-objects, the *neither-nor* structure of bi-objects might be constructed, which is opening up the *interactionality* of bridging in respect of composition and saltisition.

Hence, we have to distinguish 4 *operational* aspects of diamonds: *categorical*, *saltatorical*, *interplay* (bridge as a mix) and *interactionality* (bridging as such).

A bridge consists of a combination of acceptional and rejectional morphisms, while bridging is an abstraction from the bridge as a complex 2-object. It is the process of building a bridge, hence it is not a combination of acceptional and rejectional operations but an operation in itself.

Thus, we have to distinguish:

(composition, saltisition, combinition, bridging): [o, | |, |o|, •].

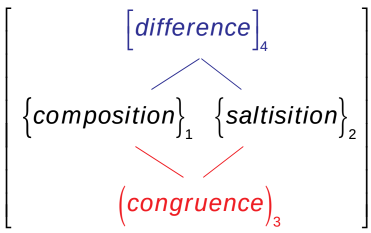
$\text{bridge}_{|o|}$: l |o| g |o| k

$\text{bridging}_{\bullet}$: l • g • k

Operative diamond

Diamond of

[*comp, salt, congruence, difference*]



The list of operation which appears in the descriptive scheme of diamonds are:

composition, saltisition, congruence and difference operations. This quadruple, again, is building a diamond structure.

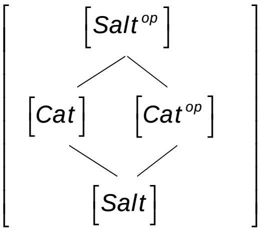
More diamonds to build

Depending on the interests of thematizing diamondization different additional diamonds might be constructed. A further study will be involved into the systematization of the different second-order diamond thematizations.

6.4.2 Architectonic approach to higher-order Diamonds

A possible distribution of the diamond [Cat, Salt] over 4 proto-structural places is given by the quadruples of categories and saltatories and their duality.

Diamond *of*
$\left[Cat, \; Cat^{op}, \; Salt, Salt^{op}\right]$



The diamond DD=[Cat, Cat^{op}, Salt, $Salt^{op}$] is a diamond unit for dissemination. Dissemination happens as iterative and accretive repetition.

As in category theory where the pattern for linear composition is a ternary composite of morphisms, for diamond theory, the basic pattern of tabular composition is the chiastic diamond with its interplay of categories and saltatories.

Hence, there are places in the kenomic grid which are occupied with chiasms and some which are not. A *place-designator* has to manage such a placing of empty and occupied places in a kenomic grid.

$$Iter\left(Diam^{(m,\ n)}\right) = Diam^{(m,\ n+1)}$$
$$Acc\left(Diam^{(m,\ n)}\right) = Diam^{(m+1,\ n)}$$
$$Acc\left(Iter\left(Diam^{(m,\ n)}\right)\right) = Iter\left(Acc\left(Diam^{(m,\ n)}\right)\right)$$

The diamond DD=[Cat, Cat^{op}, Salt, $Salt^{op}$] is a diamond unit for dissemination. Dissemination happens as iterative and accretive repetition.

Diam = [DD, Iter, Acc, pos].

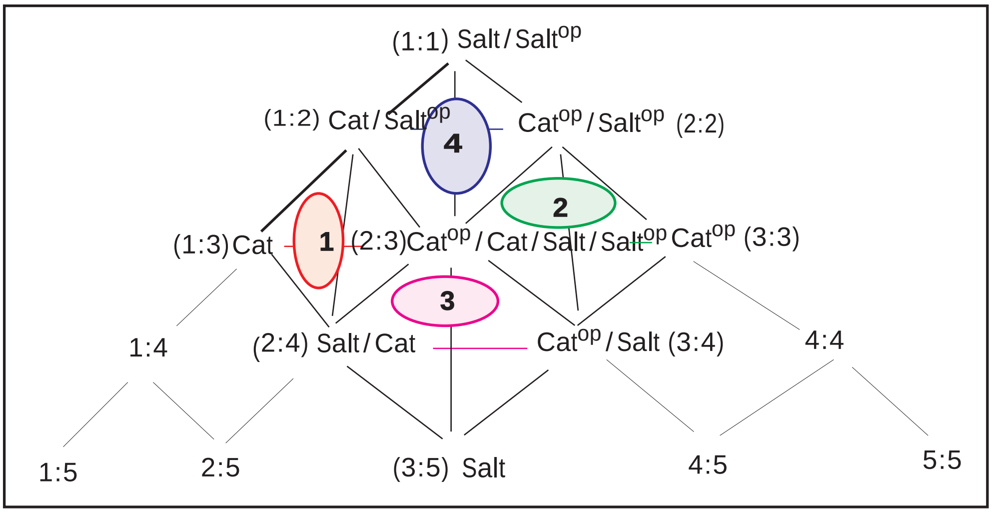
The dissemination of diamonds, organized by the place-designator, is purely structural. It disseminates diamonds independently of their contextural interpretation. Like formal logical calculi get a semantic interpretation to make them a logic, distributed diamonds needs a contextural thematization to make them a complex diamond theory.

This aspect of thematization of disseminated diamonds is not yet considered in the following paragraphs.

6.4.3 Dissemination of diamonds over the proto-structural grid

Different visualizations of the dissemination of the diamond of categories and saltatories into the proto-structural grid are proposed in this paragraph.

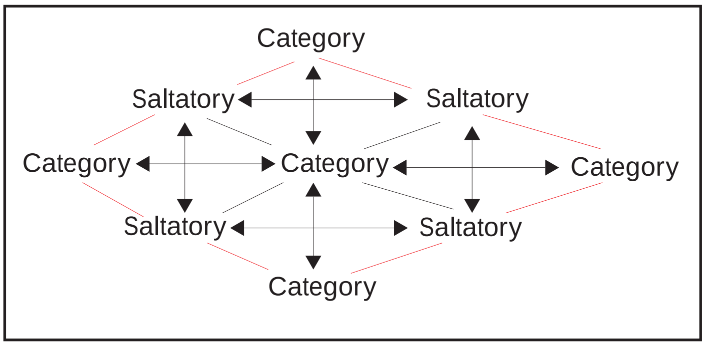
Diamonds on proto-structure




```
prop = category,Cat
opp  = dual of category, Cat^op
acc  = saltatory,Salt
rej  = dual of saltatory, Salt^op.
```

$\text{Category}_1 - \text{Category}_1{}^{op}$ / $\text{Category}_2 - \text{Category}_2{}^{op}$
$\text{Saltatory}_1 - \text{Saltatory}_1{}^{op}$ || $\text{Saltatory}_2 - \text{Saltatory}_2{}^{op}$

Simplyfied diagram



Numerical notation of the proto-structure

$$diss_{proto}^{(2,2)}\left(diam\left(Cat,\ Salt\right)\right) = \begin{bmatrix} (1:1) \\ ((1:2),\ (2:2)) \\ ((1:3),\ (2:3),\ (3:3)) \\ ((2:4),\ (3:4)) \\ (3:5) \end{bmatrix}$$

Sub-Diamond numerical notation

$$Diamond_1 = \begin{bmatrix} & (1:2) & \\ (1:3) & & (2:3) \\ & (2:4) & \end{bmatrix} \quad Diamond_2 = \begin{bmatrix} & (2:2) & \\ (2:3) & & (3:3) \\ & (3:4) & \end{bmatrix}$$

$$Diamond_3 = \begin{bmatrix} & (2:3) & \\ (2:4) & & (3:4) \\ & (3:5) & \end{bmatrix} \quad Diamond_4 = \begin{bmatrix} & (1:1) & \\ (1:2) & & (2:2) \\ & (2:3) & \end{bmatrix}$$

Interpreted numerical diamond

$$Diamond_5 = \begin{bmatrix} & (1:1) & \\ (1:3) & & (3:3) \\ & (3:5) & \end{bmatrix} = \begin{bmatrix} & Salt_5^{op} & \\ Cat_5 & & Cat_5^{op} \\ & Salt_5 & \end{bmatrix}$$

The thematization of a category as a saltatory and a saltatory as a category is ruled by the operation of complementarity. More technically, diamonds are ruled by the as-abstraction of thematization, while category theory is ruled by the is-abstraction of identification.

To take the dual of a category as a new category is ruled by the operation of accretive duality. That is, in contrast to the duality of duality rule of categories which is idempotent, the accretive duality of a duality is augmenting the complexity of the diamond system by the interchange of a dual category at a location to a new category at another augmented location. In the example, an accretion of categories is inscribed and an iteration of saltatories as two chains:

accretive chain: Cat – Cat^{op} / Cat – Cat^{op} / … / Cat – Cat^{op}

iterative chain: $Salt^{op}$ – Salt / $Salt^{op}$ – Salt / … / $Salt^{op}$ – Salt.

Cat – Salt –…–Cat – Salt

Salt – Cat –…–Cat – Salt

The grid of disseminated diamonds offers a binomial number of paths between its knots labelled as Cat, Salt, Cat^{op} and $Salt^{op}$.

It should be mentioned that the chiastic chain of diamonds is neutral to an origin or an end of its development. The same argument holds for the operation of complementarity. Again, we have to distinguish intra-diamondal complementarity which is idempotent, compl(compl(X)=X, with trans-diamondal complementarity which is iterative and accretive.

Iterative and accretive composition of diamonds

$$diss^{(3,3)}\left[Cat,\ Cat^{op},\ Salt,\ Salt^{op}\right] = \begin{bmatrix} & \left[Salt^{op}\right] & \\ & \diagup \quad \diagdown & \\ \left[Cat\right] & & \left[Cat^{op}\right] \\ & \diagdown \quad \diagup & \\ & \left[Salt\right] & \end{bmatrix}^{(3,\ 3)}$$

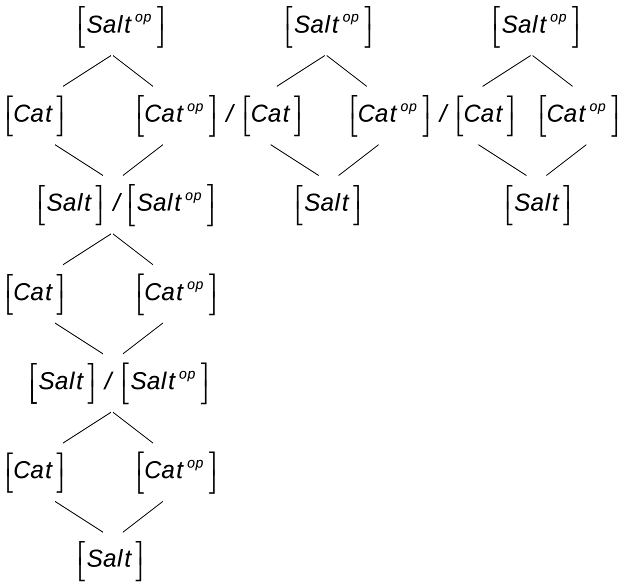
Motivation for disseminating diamonds

In Ancient Chinese mathematics it is said that a good mathematician the one which is enlarging the realm of kinds by opening up new kinds (lei) which are serving as the context in which problems find a resolution. In a modern translation it can be said that the kinds are corresponding to contextures and the inherent structure of contextures are realized by diamonds.

The other part of the structure of dissemination has its Chinese correspondence in the finiteness of kinds. This, combined with the temporal structure of happenstance, is motivating the antidromic structure of diamonds. Antidromic structures are interpreted by Jinmei as the bi-directionality of mathematical moves.

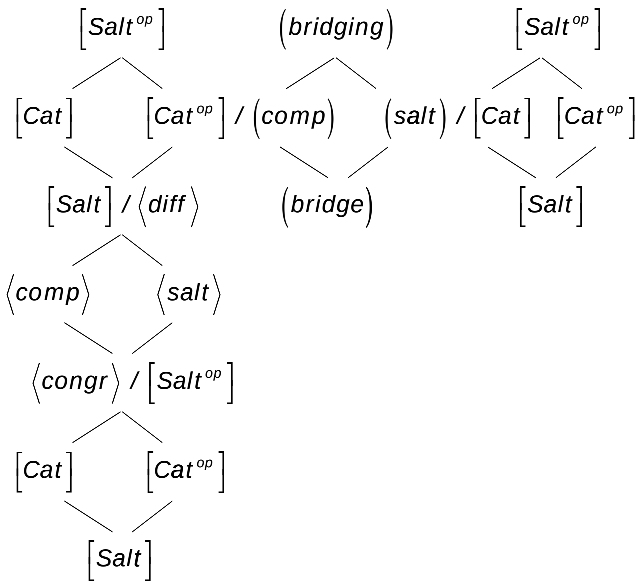
In other words, dialectical, polycontextural and Ancient way of thinking is fundamentally paradoxical, antinomic, contradictory, i.e., dialectical and not to be sublimed by a unitarian, unifying idendity force. Hence, a move forwards is at once a move backwards.

http://www.thinkartlab.com/CCR/2007/07/chinese-ontology.html



Mixing different types of diamonds

Following the distinctions of different types of diamonds, as introduced above, different patterns of dissemination of diamonds over the proto-structure might be introduced.



Hence, what is in the conceptual role of an opposite of a category appears in another diamond as the concept of composition.

And: What is in the conceptual role of a saltatory in a diamond appears in another diamond as the concept of difference.

And: What is in the conceptual role of a congruence in a diamond appears in another diamond as the concept of the opposite of a saltatory.

And: What is in the conceptual role of a saltisition in a diamond appears in another diamond as the concept of a category.

Table notation of dissemination

$diss^{(2,2)}\left(diam\left(Cat, Salt\right)\right) =$

DM	O_1	O_2	O_3
M_1	$Prop_1$	Rej_1 / $Prop_4$	Rej_4
M_2	Acc_1 / $Prop_3$	Opp_1 / $Prop_2$ / Rej_3 / Acc_4	Rej_2 / Opp_4
M_3	Acc_3	Acc_2 / Opp_3	Opp_2

Terms to be disseminated, which are building a diamond:

[Prop,Opp,Acc, Rej].

More about positioning of diamonds in the chapter *"Positionality of diamonds"*.

Duality and Complementarity Cycles

$$dual_i\left(dual_i\left(X\right)\right) = X,\ i = 1,2$$

$$dual_1\left(dual_2\left(dual_1\left(X\right)\right)\right) = dual_2\left(dual_1\left(dual_2\left(X\right)\right)\right)$$

$$comp_i\left(comp_i\left(X\right)\right) = X,\ i = 1,2$$

$$comp_1\left(comp_2\left(comp_1\left(X\right)\right)\right) = comp_2\left(comp_1\left(comp_2\left(X\right)\right)\right)$$

$$Iter\left(Diam^{(m,\ n)}\right) = Diam^{(m,\ n+1)}$$

$$Acc\left(Diam^{(m,\ n)}\right) = Diam^{(m+1,\ n)}$$

$$Acc\left(Iter\left(Diam^{(m,\ n)}\right)\right) = Iter\left(Acc\left(Diam^{(m,\ n)}\right)\right)$$

6.4.4 Compositions of Diamonds

According to the principles of polycontextural iterability, repetition has to be distinguished as *iterative* and *accretive* repetition. In classical category theory composition is of iterative nature.That is, the iteration of the operation "composition" is enclosed in its contexture, and there is no chance to leave this contexture. Hence, composition in categories is closed. The complementary aspect of iterability in polycontextural systems is accretivity. Accretive operations are leaving the contexture for another contexture, augmenting the structural complexity of the system.

As a possible proposal to an implementation of full iterability, i.e., accretivity and iterativity, for diamond systems, the following strategy is risked.

Iterability of composition

$$
\begin{array}{ccccccc}
\alpha_1 & \longrightarrow & \omega_1 & \Leftrightarrow^{iter} & \alpha_1{}' & \longrightarrow & \omega_1{}' \\
\Updownarrow^{acc} & & \Updownarrow^{acc} & & \Updownarrow^{acc} & & \Updownarrow^{acc} \\
\omega_2 & \longleftarrow & \alpha_2 & \Leftrightarrow^{iter} & \omega_2{}' & \longleftarrow & \alpha_2{}'
\end{array}
$$

$$
\begin{array}{ccccccc}
\alpha_1 & \longrightarrow & \omega_1 & \Leftrightarrow^{iter} & \alpha_1{}' & \longrightarrow & \omega_1{}' \\
 & & \Updownarrow^{acc} & & \Updownarrow^{acc} & & \\
 & & \alpha_2 & \Leftrightarrow^{iter} & \omega_2{}' & &
\end{array}
$$

To each order relation (morphism, arrow) a *double exchange* relation is attached, the iterative and the accretive exchange relation.

To show the essentials of the double-exchange relations, this graph is omitting the additional properties of the diamond, i.e., the coincidence relations and the acceptional and rejectional morphisms of the full diamond structure.

This structure of complex iterativity for categories was never studied in detail before. But it was introduced, informally in my papers, as iterative and accretive grids of chiasms, long ago. No precise mechanism of complex composition was given at that time. For polycontextural logics and contextural programming, tabularity was developed to some extend.

Thus, this construction risked now has to be regarded as a very first step of introducing accretivity and iterativity into the rules of morphism composition. This construction is obviously based on the functional distinction of alpha- and omega-properties of morphisms. It seems not to be naturally accessible with the classic definition of categorical objects alone. Nor is it simply a kind of products of categories, say fibred categories or similar, which had been used to formalize polycontextural logics (Pfalzgraf).

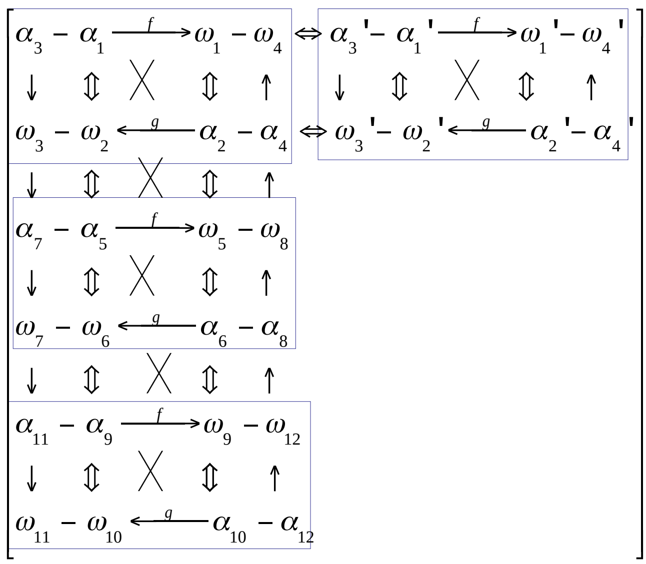
Tabular dissemination of diamonds happens on the very base of their definition, and not as a secondary construction. This, surely, is in no way excluded by basic dissemination of diamonds.

$$\coprod_{m,\,n}\left[\mathbf{A};\ \mathbf{a}\right]^{(m,\,n)} = \left(\begin{array}{ccc} \left[\mathbf{A};\ \mathbf{a}\right]\odot\left[\mathbf{A};\ \mathbf{a}\right]\odot,\ldots, & & \odot\left[\mathbf{A};\ \mathbf{a}\right] \\ \oplus & & \oplus \\ \left[\mathbf{A};\ \mathbf{a}\right] & & \left[\mathbf{A};\ \mathbf{a}\right] \\ \oplus & & \oplus \\ . & & . \\ \oplus & & \oplus \\ \underbrace{\left[\mathbf{A};\ \mathbf{a}\right]\odot\left[\mathbf{A};\ \mathbf{a}\right]\odot,\ldots,\odot\left[\mathbf{A};\ \mathbf{a}\right]}_{s(m,\,n)} & & \end{array} \right)$$

Block diagrams for diamond grids

The notation of the chiastic composition structure can be omitted by the block representation of the composition of the basic chiasms. Hence, the bracket are symbolizing chiastic composition at all of their 4 sides, left / right and top / bottom. That is, the top and bottom aspects are representing chiastic compositions in the sense of accretion of complexity. The right / left-aspects are connections in the sense of iterative complication. Iteration per se is not chiastic but compositional in the usual sense.

Accretive and mixed iterative+accretive iterability



Iterative composition is coincidental, accretive composition is chiastic. Coincidental composition is based on the coincidence of domains and codomains of morphisms, chiastic composition is based on the exchange relation between alpha and omega properties of morphisms. Both together, are defining the free composition of diamonds. This wording might be misleading if we consider the introduction of two types of exchange relations, the accretive and the iterative.

6.4.5 Duality of Diamonds

Duality for Categories

"The concept of category is well balanced, which allows an economical and useful duality. Thus in category theory the "two for the price of one" principle holds: every concept is two concepts, and every result is two results." (Herrlich)

"The **Duality Principle for Categories** states

Whenever a property P holds for all categories,
then the property P^{op} holds for all categories.

The proof of this (extremely useful) principle follows immediately from the facts that for all categories A and properties P
(1) $(\mathbf{A}^{op})^{op} = \mathbf{A}$, and
(2) $P^{op}(\mathbf{A})$ holds if and only if $P(\mathbf{A}^{op})$ holds." (Herrlich)

THE DUALITY PRINCIPLE

3.5 DEFINITION

For any category $\mathbf{A} = (\mathcal{O}, \hom_{\mathbf{A}}, id, \circ)$ the **dual** (or **opposite**) **category of A** is the category $\mathbf{A}^{op} = (\mathcal{O}, \hom_{\mathbf{A}^{op}}, id, \circ^{op})$, where $\hom_{\mathbf{A}^{op}}(A, B) = \hom_{\mathbf{A}}(B, A)$ and $f \circ^{op} g = g \circ f$. (Thus $\mathbf{A}$ and $\mathbf{A}^{op}$ have the same objects and, except for their direction, the same morphisms.)

Duality for Saltatories

Obviously, *jumpoids* in diamonds are not the dual of a category. Simply because they are not categories but jumpoids, not being defined in the same way as categories.

But diamonds can have duals. Different strength of duality of diamonds, categories and jumpoids, can be introduced.

The dualization of a category is a dual category, thus, still a category.

A dualization of a jumpoid is dualizing its category, and vice versa, a dualization of a category in a diamond is dualizing its jumpoid. A dualization of a diamond is a dualization of its categories and its jumpoids together.

Duality in Diamonds

$X = g \lozenge f = \left[\left(g \; o \; f\right); \; u\right]$

1. $X \in Cat \;\; iff \; dual\left(X\right) \in Cat^{op}$

$\left(g \; o \; f\right) = A \to C$

$$dual\left(g \; o \; f\right) = dual \; \left(dual\left(B \to C\right) \; o \; dual\left(A \to B\right)\right)$$
$$= dual \; \left(\left(B \leftarrow C\right) \; o \; \left(A \leftarrow B\right)\right)$$
$$= \; \left(\left(A \leftarrow B\right) \; o \; \left(B \leftarrow C\right) \; \right)$$
$$= \left(A \leftarrow B \leftarrow C\right)$$
$$= A \leftarrow C.$$

$Hence, \; \left(\left(g \; o \; f\right) = A \to C\right) \in Cat \;\; iff \;\; \left(dual\left(g \; o \; f\right) = A \leftarrow C\right) \in Cat^{op}.$

2. $X \in Salt \;\; iff \; dual\left(X\right) \in Salt^{op}$

$u = \left(\omega_4 \leftarrow \alpha_4\right) = compl\left(g \; o \; f\right)$

$dual\left(compl\left(g \; o \; f\right)\right) = dual\left(u\right)$

$$dual\left(u\right) \;\; = dual\left(\omega_4 \leftarrow \alpha_4\right)$$
$$= \left(\alpha_4 \to \omega_4\right).$$

$compl\left(dual\left(g \; o \; f\right)\right) = compl\left(f \; o \; g\right) = \left(\alpha_4 \to \omega_4\right).$

$Hence, \; \left(u = \left(\omega_4 \leftarrow \alpha_4\right)\right) \in Salt \;\; iff \;\; \left(dual\left(u\right) = \alpha_4 \to \omega_4\right) \in Salt^{op}.$

Duality in Diamonds

duality in categories	*duality in saltatories*
$\left(g \; o \; f\right) = A \to C$ $dual\left(g \; o \; f\right) = dual \; \left(dual\left(B \to C\right) \; o \; dual\left(A \to B\right)\right)$ $= dual \; \left(\left(B \leftarrow C\right) \; o \; \left(A \leftarrow B\right)\right)$ $= \; \left(\left(A \leftarrow B\right) \; o \; \left(B \leftarrow C\right) \; \right)$ $= \left(A \leftarrow B \leftarrow C\right)$ $= A \leftarrow C.$ $Hence, \; \left(\left(g \; o \; f\right) = A \to C\right) \in Cat$ *iff* $\left(dual\left(g \; o \; f\right) = A \leftarrow C\right) \in Cat^{op}.$	$u = \left(\omega_4 \leftarrow \alpha_4\right) = compl\left(g \; o \; f\right)$ $dual\left(compl\left(g \; o \; f\right)\right) = dual\left(u\right)$ $dual\left(u\right) = dual\left(\omega_4 \leftarrow \alpha_4\right)$ $= \left(\alpha_4 \to \omega_4\right).$ $compl\left(dual\left(g \; o \; f\right)\right) = compl\left(f \; o \; g\right) = \left(\alpha_4 \to \omega_4\right).$ $Hence, \; \left(u = \left(\omega_4 \leftarrow \alpha_4\right)\right) \in Salt$ *iff* $\left(dual\left(u\right) = \alpha_4 \to \omega_4\right) \in Salt^{op}.$
$X = g \lozenge f = \left[\left(g \; o \; f\right); \; u\right]:$	
$X \in Cat \;\; iff \; dual\left(X\right) \in Cat^{op}$	$X \in Salt \;\; iff \; dual\left(X\right) \in Salt^{op}$

Duality for Diamonds

$$X = g \lozenge f = \left[\left(g \ o \ f\right); \ u\right]$$
$$dual\left(X\right) = dual\left(g \lozenge f\right)$$
$$= dual\left(dual\left(g\right) \lozenge dual\left(f\right)\right)$$
$$= \left(dual\left(f\right) \lozenge dual\left(g\right)\right)$$
$$= \left[\left(dual\left(f\right) \ o \ dual\left(g\right)\right); \ dual\left(u\right)\right]$$

$$For \ \left[\mathbf{A} = category; \ \mathbf{a} = saltatory\right]:$$
$$\left[\mathbf{A}; \ \mathbf{a}\right] \in Diam \ \ iff \ \ dual\left(\left[\mathbf{A}; \ \mathbf{a}\right]\right) \in Diam^{op}$$
$$iff$$
$$\begin{bmatrix} \mathbf{A} \in Cat \ \ iff \ \ dual\left(\mathbf{A}\right) \in Cat^{op} \\ \mathbf{a} \in Sal \ \ iff \ \ dual\left(\mathbf{a}\right) \in Salt^{op} \end{bmatrix}.$$

$$\left[\mathbf{A}; \ \mathbf{a}\right] \in Diam \ \ iff \ \ dual\left(dual\left(\left[\mathbf{A}; \ \mathbf{a}\right]\right)\right) \in Diam$$
$$dual\left(P\right)\left(\left[\mathbf{A}; \ \mathbf{a}\right]\right) \ \ iff \ \ P\left(dual\left(\left[\mathbf{A}; \ \mathbf{a}\right]\right)\right)$$
$$= P\left(dual\left(\left[dual\left(\mathbf{A}\right), \ dual\left(\mathbf{a}\right)\right]\right)\right)$$
$$\left(P^{op}\right)\left(\left[\mathbf{A}; \ \mathbf{a}\right]\right) \ \ iff \ \ P\left(\left[\mathbf{A}; \ \mathbf{a}\right]^{op}\right) = P\left(\left[\mathbf{A}^{op}; \ \mathbf{a}^{op}\right]\right)$$

$$\left[Cat; \ Salt\right] \in Diam:$$
$$dual\left(Cat\right) \ \ iff \ \ dual\left(Salt\right)$$

$$\left[\mathbf{A}; \ \mathbf{a}\right] \in Diam:$$
$$dual\left(\left[\mathbf{A}; \ \mathbf{a}\right]\right) = \left[dual\left(\mathbf{A}\right); \ dual\left(\mathbf{a}\right)\right]$$

$$\left[\mathbf{A}; \ \mathbf{a}\right] \in Diam: \left[\mathbf{A}; \ \mathbf{a}\right]^{op} = \left[\mathbf{A}^{op}; \ \mathbf{a}^{op}\right]$$

Diamonds are not elements of the "periodic" system of n-categories.

6.4.6 Complementarity of Diamonds

Complementarity is a feature of the interplay between categories and saltatories.

Between acceptional and rejectional configurations a complementarity is involved.

As much as duality is an important principle of category theory the corresponding transversal principle of complementarity is of the same importance. The complementarity principle for diamonds is a new property of formal systems unknown to category theory.

Complementarity and duality

The interplay of duality and complementarity get a more intricate picture if we introduce partial dualities and partial complementarities.

More general: **Categorification and Diamondization**.
[(Categorification, Diamondization), Dissemination]
The two main trans-classical strategies are: dissemination and diamondization.

The Diamond was introduced as a complexion of 4 basic properties:

1. proposition,
2. opposition,
3. acceptance,
4. rejectance.

The relationship between those diamond properties and the categorial definition of the diamond is re-established by the equations for *acceptance* and *rejectance* relative to their morphisms.

complementarity of accept, reject

$$reject(gf) = k \ \ iff \ \ accept(k) = (gf)$$
$$reject(hg) = l \ \ iff \ \ accept(l) = (hg)$$
$$reject(hgf) = m \ \ iff \ \ accept(m) = (hgf)$$

lThus, the operation *reject(gf)* of the the acceptance morphisms f and g is producing the rejectance morphism k.
And the operation *accept(k)* of the rejectance morphism k is producing the acceptance of the morphisms g and f.

The acceptance of f*g, acc(f,g), is the *composition* of f and g, (f o g).

The rejectance of f*g, rej(f,g) is the *hetero-morphism* of f and g, (g^{o},f^{o}).

The acceptance of f*g*h, acc(f,g,h), is the *composition* of f, g and h, (fgh).

The rejectance of f*g*h, rej(f,g,h) is the *jump* morphism of f^{a} and h^{o}, (h^{o},f^{o}).

The acceptance f^{a} and h^{o}, acc(h^{o},f^{o}) is the *spagat* of f^{o} and h^{o}, ($f^{o}h^{o}$).

The acceptance f^{a}, g and h^{o}, acc(h^{o},g, f^{o}) is the *bridge* g of f^{o} and h^{o}, ($f^{o}gh^{o}$).

Diamond

category	saltatory
objects	*abjects*
morph	*hetero – m*
identity	*difference*
composition	*jump*
bridge	*spagat*
duality	*compl*

Diamond - Category DC

Category : $\mathbf{A} = \left(Obj^{\mathbf{A}},\ hom,\ id,\ o\right)$

Jumpoid : $\mathbf{a} = \left(Obj^{a},\ het,\ id,\ \|\right)$

DC $= \left(\left[\mathbf{A};\ \mathbf{a}\right],\ compl,\ diff,\ \bullet\right)$

Interactivity in diamonds / diamonds of interactions

Essential for the definition of the *category* is the composition operation and its associativity. Associativity enters the game with the composition of 3 morphisms.

In the same way, the definition of *diamonds* is ruled by the diamond composition and the necessity of 4 morphisms.

A composition in a *category* is defined by the coincidence of the codomain *cod* and the domain *dom* of the composed morphisms.

A composition in a *diamond* has always to reflect additionally the difference, i.e., the *complement* of the categorical composition operation. Thus, a diamond composition is producing a *composite* and a *complement* of the composed morphisms. The composite is the *acceptional*, and the complement the *rejectional* part of the diamond operation.

Skeleton

Not very surprisingly, the whole story of diamond category theory begins with a 4-diamond category.

The 3-diamond category is a reduction delivering the seminal idea of a new topic in category theory and the common category is a genuine part of the 4-diamond.

The 3-diamond, categorial or as conceptual graph, is introducing the new, 4th theme, giving it a position in the conceptual framework but it is not yet offering any formal laws of it, like it happens for ordinary categories. This well positioned new theme with its localisation in the kenomoc grid is characterized in 3-diamonds only up to the counter-direction of its new morphism. There is no possibility given in a 3-diamond to further characterize the laws of this counter-morphism. It is as it is, a singularity, based on a category, focusing on the difference possible in its composition laws. That is, elucidating the possible difference in / of the necessary coincidence of codomain and domain in a composition of morphisms.

Formal laws of the new theme of diamonds enter the game only for m≥4, that is the story has to start with 4-diamonds. A proper definition of associativity for counter-morphisms (hetero-morphisms) occur only for a m-diamond, m≥5. That is a composition of diamonds.

– *Categories* are dealing with morphism, identity and composition.

– *Jumpoids* are dealing with hetero-morphism, difference and jumps.

– *Diamonds* are dealing with interaction of categories and jumpoids.

Both, categories and jumpoids, are in some respect complementary but not dual.

A full 4-diamond is a mediation of two categories and one jumpoid.

What are the complementary morphisms for?

The 2-level definition of the diamond composition as a composition and a complement, opens up the possibility to control the fulfilment of the conditions of coincidence of the categorial composition from the point of view of the complementary level.

If the morphism *l* is verified, then the composition *(f o g)* is realized. The verification is checking at the level *l* if the coincidence of cod(f) and dom(g), i.e., *cod(f)=dom(g)*, for the composition "o", is realized.

Thus, simultaneously with the realization of the composition, the complementary morphism *l* is controlling the (logical, categorical) adequacy of the composition (fg).

Diamonds are involved with bi-objects. Objects of the category and counter-objects of the *jumpoid* of the diamond. Both are belonging to different contextures, thus being involved with 2 different logical systems. The interplay between categories and jumpoids is ruled by a third, mediating logic for both.

6.4.7 Complementarity – formal exposition

compl(Diamond)

$$\begin{aligned}&For\ \ \forall X \in Comp:\\ &X \in Acc\ \ iff\ \ compl\left(X\right) \in Rej,\\ &compl\left(compl\left(X\right)\right) = X.\end{aligned}$$

For all compositions X, X is an element of the acceptional domain Acc iff the complement of X, compl(X), is an element of the rejectional domain Rej.

In a strict sense there is no complementation to a single morphism. There may be a duality but no complementarity. For that, there is also no complement of a categorial object in a saltatary. For technical reasons it could be argued that the complementarity of a morphism in a category is an object in a saltatory.

Complementarity of a single morphism, not involved into composition, can be defined on the base of its own operations: domain, codomain and identity.

$$\begin{aligned}&\left[\mathbf{A};\ \mathbf{a}\right] \in Diam\\ &iff\\ &compl\left(\left[\mathbf{A};\ \mathbf{a}\right]\right) = \left[\mathbf{a};\ \mathbf{A}\right] \in Diam\end{aligned}$$

$$\begin{aligned}&revrs\left(\left[\mathbf{A};\ \mathbf{a}\right]\right) \in Diam\\ &iff\\ &\forall x \in \mathbf{A},\ \forall y \in \mathbf{a}\\ &\left[compl\left(\mathbf{A}\right);\ compl\left(\mathbf{a}\right)\right] \in Diam\end{aligned}$$

The complement of a categorical morphism can be introduced by the "trick" of using the identity operation id:

$$f\ :\ A \rightarrow B,\ id_A\ o\ f = f = f\ o\ id_B$$
$$\Rightarrow \overline{id_A\ o\ f} = \overline{f} = \overline{f\ o\ id_B}$$

$$1.\ \overline{f} = compl\left(id_A\ o\ f\right)$$
$$\overline{f} = compl\left(compl\left(id_A\right)\ o\ compl\left(f\right)\right)$$
$$\overline{f} = compl\left(diff\left(A\right)\ o\ compl\left(f\right)\right)$$
$$\overline{f} = compl\left(diff\left(A\right)\ o\ compl\left(A \rightarrow B\right)\right)$$
$$\overline{f} = compl\left(diff\left(A\right)\ o\ \left(diff\left(A\right) \leftarrow diff\left(B\right)\right)\right)$$
$$\overline{f} = \left(diff\left(A\right) \leftarrow \left(diff\left(A\right) \leftarrow diff\left(B\right)\right)\right)$$
$$\overline{f} = \left(diff\left(A\right) \leftarrow diff\left(B\right)\right)$$
$$\overline{f} = \left(\overline{A}\right) \leftarrow \left(\overline{B}\right)$$

$$2.\ \overline{f} = compl\left(f\ o\ id_B\right)$$
$$\overline{f} = compl\left(compl\left(f\right)\ o\ compl\left(id_B\right)\right)$$
$$\overline{f} = compl\left(compl\left(f\right)\ o\ diff\left(B\right)\right)$$
$$\overline{f} = compl\left(compl\left(A \rightarrow B\right)\ o\ diff\left(B\right)\right)$$
$$\overline{f} = \left(\left(diff\left(A\right) \leftarrow diff\left(B\right)\right) \leftarrow diff\left(B\right)\right)$$
$$\overline{f} = \left(diff\left(A\right) \leftarrow diff\left(B\right)\right)$$
$$\overline{f} = \left(\overline{A}\right) \leftarrow \left(\overline{B}\right)$$
$$Hence,\ \left(1.\right) = \left(2.\right) = \overline{f}$$

The complement of a *right*-identity of A is a *left*-identity over the complement of A, $\overline{A}$. Thus, complementarity of objects for categories and saltatories is identical with the change in direction of the identity operation. Such a property is of no meaning for categories alone. The new properties for objects, i.e., bi-objects, are identity, diversity, left, right.

$$\left.\begin{array}{l} f : A \to A \\ id_A \ o \ f = f \end{array}\right\rangle : \left(\bar{f}\right) = \overline{id_A \ o \ f}$$

$$\begin{aligned}
\bar{f} &= compl\left(id_A \ o \ f\right) \\
\bar{f} &= compl\left(compl\left(id_A\right) \ o \ compl\left(f\right)\right) \\
\bar{f} &= compl\left(diff\left(A\right) \ o \ compl\left(f\right)\right) \\
\bar{f} &= compl\left(diff\left(A\right) \ o \ compl\left(A \to A\right)\right) \\
\bar{f} &= compl\left(diff\left(A\right) \ o \ \left(diff\left(A\right) \leftarrow diff\left(A\right)\right)\right) \\
\bar{f} &= \left(diff\left(A\right) \leftarrow \left(diff\left(A\right) \leftarrow diff\left(A\right)\right)\right) \\
\bar{f} &= \left(diff\left(A\right) \leftarrow diff\left(A\right)\right) \\
\bar{f} &= \left(\bar{A}\right) \leftarrow \left(\bar{A}\right)
\end{aligned}$$

The complement of the binary composition (g o f), is the *hetero-morphism* u.

Complementarity of Acc and Rej

$X \in Acc \ iff \ compl\left(X\right) \in Rej$

$X = g \ o \ f :$

1. $X \in Acc \ if \ compl\left(X\right) \in Rej$

$$\begin{aligned}
compl\left(g \ o \ f\right) &= compl\ \left(compl\left(g\right) \ o \ compl\left(f\right)\right) \\
&= compl\left(diff\left(cod\left(f\right)\right) \ o \ diff\left(dom\left(g\right)\right)\right) \\
&= compl\left(\left(\overline{B_{cod}}\right) \ o \ \left(\overline{B_{dom}}\right)\right) = \omega_4 \leftarrow \alpha_4 .
\end{aligned}$$

$\left(u : \omega_4 \leftarrow \alpha_4\right) \in Rej$

$Hence, \ \left(g \ o \ f\right) \in Acc \ if \ \left(u : \omega_4 \leftarrow \alpha_4\right) \in Rej$

$\left(g \ o \ f\right) \in Acc \ if \ \left(\overline{g \ o \ f}\right) \in Rej.$

2. $compl\left(X\right) \in Rej \ if \ X \in Acc$

$$\begin{aligned}
compl\left(\omega_4 \leftarrow \alpha_4\right) &= compl\left(compl\left(\omega_4\right) \leftarrow compl\left(\alpha_4\right)\right) \\
&= compl\left(\left(A_{dom} \to B_{cod}\right) \leftarrow \left(B_{dom} \to C_{cod}\right)\right) \\
&= \left(\left(A_{dom} \to B_{cod}\right) \ o \ \left(B_{dom} \to C_{cod}\right)\right) \\
&= \left(f \ o \ g\right).
\end{aligned}$$

3. $Hence, \ X \in Acc \ iff \ compl\left(X\right) \in Rej.$

In this "proof", the complementarity operation "compl" is used quite freely to do also the transitional job of completing the morphisms out of the objects. This is done by the operation of "difference" and "completition", which is completing domains and codomains to their morphisms. This points to the asymmetry of Acc- and Rej-domains.

The complement of a ternary composition (f o g o h) is a *jumpoid* (u| |v).

$$\left(f\ o\ l\ o\ m\right) \in Acc \ \ iff\ \ \left(\overline{f\ o\ l\ o\ m}\right) \in Rej$$
$$\left(f\ o\ l\ o\ m\right) = h$$
$$rej(h) \ = \ (u \parallel v) \in Rej$$
$$acc(u \parallel v) = h \in Acc$$

The operators *acc* and *rej* are specifications of the general operator "compl" of complementation.

Duality between categories is symmetrical and thus preserving complexity of a situation. Complementarity for diamonds is establishing an asymmetry between categories and saltatories. Saltatories of categories are of lower complexity (complication) than their complementary categorical parts they are representing by complementation. In this sense, saltatories are abstractions from categories.

Complementarity between morphisms and hetero-morphisms

A new kind of complementarity has to be considered. The complementarity between the morphism g and the hetero-morphism m.

The morphism *g* is understood as an inter-mediate morphism between morphisms *f* and *h*, i.e., (*f o g o h)*.

The complements of *(f o g o h)* are the hetero-morphisms *l*, *m*, composed in the jump-composition *(k| |)= m*.

The direct complement or opposite to the morphism g is compl(g) = m.

In the same sense as (f o g) =h, compl(f o g)= l, the rejectional opposite of h is l.

6.4.8 Complementarity and Duality

Duality in category theory

Duality in category theory is mainly conceived as an economic tool. It serves to get "2 for 1" in the business of proofs as Herrlich says: *"Thus in category theory the "two for the price of one" principle holds: every concept is two concepts, and every result is two results."* Nevertheless, category theory itself is not a 2-concept theory as such.

The duality principle might have a meta-theoretical function, too and is proof of the symmetric beauty of category theory. But it doesn't serve as a conception for more complex tasks. That is, the duality as such is not a specific topic of category theory. It may be a topic for n-categorical studies, but this is another story. Category theory is dealing with categorical constructions and their theorems inside the realm of a category. Everything developed there has its dual part. And this fact can be exploited to shorten proofs. But the dual part of categorical properties is not of any special interests, simply because it is not telling more than the dual to what we know already. That is, the properties P remain the same in both settings.

Hence it wouldn't make much sense to develop a dual category theory *ab origine*, i.e., a dual category theory, where both side of the duality would be developed in parallel. The laws and properties of the dual part of a category are not delivering new insights. The duality is symmetric. And where it isn't symmetric enough, techniques are applied to force it. A good example of this general strategy is given by the partly forced duality of *algebraic* and *coalgebraic* notions.

Duality for Categories
(1) $(A^{op})^{op} = A$, and
(2) $P^{op}(A)$ holds iff $P(A^{op})$ holds.

Does the 2. duality law holds for diamonds?

$$\begin{aligned}&For\ P = Assoz:\\ &A \in Assoz \Leftrightarrow A^{op} \in Assoz\\ &P(A)\ iff\ P(A^{op})\end{aligned}$$

Complementarity in diamond theory

On the other hand, complementarity in diamond theory is defined on the object-language level and its structure is not a simple symmetry.

to stay in familiar terms, complementarity operation is a new abstraction additional to the composition abstraction. Composition abstraction belongs to the family of application in combinatory logic and synthesis in the lambda calculus.

$$\begin{aligned}&\forall X \in Diam:\\ &\mathrm{dual}(\mathrm{dual}(X)) = (X)\\ &\mathrm{compl}(\mathrm{compl}(X)) = (X)\\ &compl(dual(X)) = dual(compl(X))\end{aligned}$$

6.5 Interactionality in Diamonds

Interactionality of diamonds is studying the interaction between categories and saltatories. Taken in separation, topics like complementarity are interactional, but are not yet considering the inert wining and intervening properties of interactivity.

One main property of interaction between categories and saltatories in diamonds is introduced by the operation of *bridging*. Bridging is not an operation of mediation or switching but an operation to knot two realms together, the categorical and the saltatorical. Between the hetero-morphism k, l, the morphism g is offering a bridge, marked in red, and thus interacting between the saltatorical and the categorical domain of the diamond. Complementary, the two bridge pillars of the bridge are offered by the two hetero-morphisms l, k defining the bridge work g.

Metaphor of bridge building

Thus as a metaphor, we can use the terminology of bridge building: bridge pier (pillar) and bridge are build by bridging; bridging is building the bridge and its bridge pier. There is no bridge work without bridge and its bridge pier; and there is no bridge and bridge pier without its bridge work.

What is the difference between the concept of "bridge" and "bridging"?

Similar as between composition and saltisition, which can be translated into each other by complementation, i.e. *acc* and *rej* operations, the operators *bridge* and *bridging* are translatable into each other by the operation "difference". This translation is operationally very simple, nevertheless it seems that the distinction between both concepts are well defining the double-face aspects of bridging and bridge.

Bridging is conceived from the point of view of saltatories towards categories, and the bridge is perceived from the position of categories towards saltatories.

Double - face of Bridging

$$g \bullet \left(l \parallel k\right) \in BC,$$

$$\left(l \parallel k\right) \perp g \in \widehat{BC}:$$

$$g \bullet \left(l \parallel k\right) \triangleq \left(l \parallel k\right) \perp g$$

Bridges are bridging the abyss between categories and saltatories.

Complementarity is defining the structural relationship between both.

Metaphor of movements

Two separated backward movements are supporting one movement forwards; one movement forwards is enabling two distinguished moves backwards.

One movement backwards origins in the future; the other movement backwards is oriented to the past; both are connected with the present movement forwards. One arrives from the future; one departs to the past. The movement of the present is directed to the future and distancing from the past.

This metaphor is related to morphisms, domains and codomains with the diamond difference in play but not to the composition of morphisms and their complementation as a hetero-morphism.

Bridging Conditions and Associativity for Interactions

Bridge and Bridging Conditions BC

1. $\forall k, l, n \in HET, \forall f, g, h \in MORPH:$

a. composition

$g \circ f, g \circ h,$

$(h \circ g) \circ f, h \circ (g \circ f) \in MC,$

b. saltisition

$l \parallel k, n \parallel l,$

$n \parallel (l \parallel k), (n \parallel l) \parallel k \in \overline{MC},$

c. bridges

$g \perp k, l \perp g,$

$(l \perp g) \perp k, l \perp (g \perp k)$ *are in* $\widehat{BC}.$

d. bridging

$g \bullet k, l \bullet g,$

$(l \bullet g) \bullet k, l \bullet (g \bullet k)$ *are in* $BC.$

2. $(g \bullet k) \in BC \text{ iff } dom(k) = diff(dom(g)),$

$(l \bullet g) \in BC \text{ iff } cod(l) = diff(cod(g)),$

$(l \bullet g \bullet k) \in BC \text{ iff } (g \bullet k), (l \bullet g) \in BC.$

3. $(g \perp k) \in \widehat{BC} \text{ iff } diff(dom(k)) = dom(g),$

$(l \perp g) \in \widehat{BC} \text{ iff } diff(cod(l)) = cod(g),$

$(l \perp g \perp k) \in \widehat{BC} \text{ iff } (g \perp k), (l \perp g) \in \widehat{BC}.$

Bridging

Assoziativity :

If $k, g, l \in BC$, *then* $(k \bullet g) \bullet l = k \bullet (g \bullet l),$

Bridging :

$bridging_{(g, l, k)} : het(\omega_4, \alpha_4) \bullet hom(\alpha_2, \omega_2) \bullet het(\omega_8, \alpha_8) \rightarrow het(\omega_9, \alpha_9).$

Bridge

Assoziativity :

If $k, g, l \in \widehat{BC}$, *then* $(k \perp g) \perp l = k \perp (g \perp l),$

Bridge :

$bridge_{(g, l, k)} : het(\omega_4, \alpha_4) \perp hom(\alpha_2, \omega_2) \perp het(\omega_8, \alpha_8) \rightarrow het(\omega_9, \alpha_9).$

Bridges vs. Bridging vs. Jumping

$(l \perp g \perp k) \triangleq (l \bullet g \bullet k) \triangleq (l \parallel k),$

$(l \perp g \bullet k) \triangleq (l \bullet g \perp k) \triangleq (l \parallel k),$

$(l \bullet g \perp k) \triangleq (l \perp g \bullet k) \triangleq (l \parallel k).$

$diff(\perp) = (\bullet), (\perp) = diff(\bullet).$

Bridging Conditions BC are the matching conditions MC for bridging mappings.

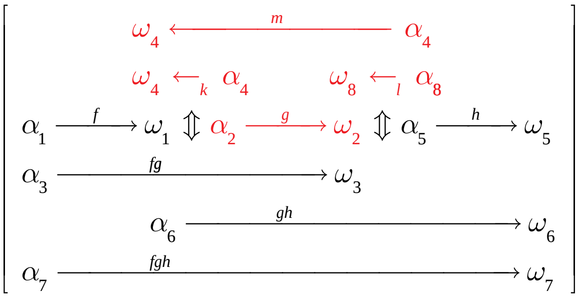
Proofs for interactionality (cf. distributivity)

Collecting terms

Category: composition based on matching conditions (coincidence)
Saltatory: saltisition based on jumping conditions
Interactionality: bridge, bridging, transversality, parallelity based on bridging conditions (difference).

Possible chain of operators

composition (o) produced by morphisms, matching condition, domain, codomain,
saltisition (| |) produced by complementation (difference) of composition,
bridge (^) produced by composition and difference from category and saltatory,
bridging (•) produced by difference from bridge.



As a consequence, the composition (f o g) and the saltisition (k | | l) are mixed to (l | | k) o g).

Bridging vs. jumping shows clearly that not only *what* is achieved matters but *how* it is achieved, i.e., by bridging or by jumping. Each jump in a saltatory has an inverse morphism as a bridge in a category. Or, *rej(g)=m* and *acc(m)=g*.

Duality for Bridging

$$X = (f \bullet k),\ f : A \to B,\ k : \omega_4 \leftarrow \alpha_4 :$$
$$\begin{aligned} dual(f \bullet k) &= dual(dual(f) \bullet dual(k)) \\ &= dual((B \to A) \bullet (\alpha_4 \to \omega_4)) \\ &= (\alpha_4 \to \omega_4) \bullet (B \to A). \end{aligned}$$

$$X = (g \bullet l) \,||\, (g \bullet k),\ g : B \to C,\ l : \omega_4 \leftarrow \alpha_4,\ k : \omega_8 \leftarrow \alpha_8 :$$
$$\begin{aligned} dual(X) &= dual(dual(g \bullet l) \,||\, dual(g \bullet k)) \\ &= dual(g \bullet k) \,||\, dual(g \bullet l) \\ &= (dual(k) \bullet dual(g)) \,||\, (dual(l) \bullet dual(g)) \\ &= (dual(l) \bullet dual(g)) \,||\, (dual(k) \bullet dual(g)) \\ &= (dual(g) \bullet dual(l)) \,||\, (dual(g) \bullet dual(k)) \\ &= dual(g \bullet l) \,||\, dual(g \bullet k). \end{aligned}$$

6.5.1 Distributivity for Interactions

Distributivity for Sets

"Given a set S and two binary operations • and + on S, we say that
• is left-distributive over + if, given any elements x, y, and z of S,

x • (y + z) = (x • y) + (x • z);

• is right-distributive over + if, given any elements x, y, and z of S:

(y + z) • x = (y • x) + (z • x);

• is distributive over + if it is both left- and right-distributive.
Notice that when • is commutative,
then the three above conditions are logically equivalent."

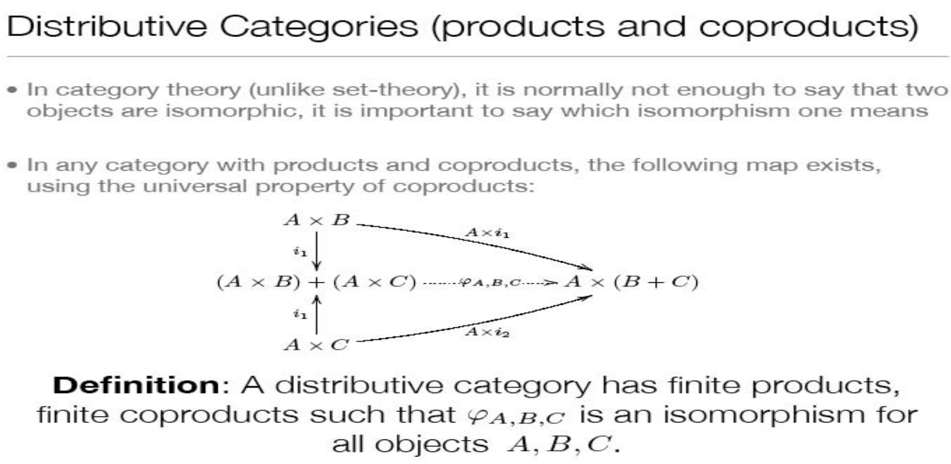
http://en.wikipedia.org/wiki/Distributivity

Distributivity in Category theory

Distributivity occurs in category theory as the distributivity of *products* and *coproducts*. In the basic definition of category itself there is no space for such a definition of distributivity, simply because there is one and only one operator involved: *composition*. And to realize distribution, at least two operators are necessary. Compositions are commutative, identive and associative; but not distributive. Categoricity of category theory is highly abstract and is reducing operationality to the single operation of composition. Compositionality is the concept and the operation of categories.

Diversity enters into the formalism with the category-based constructions of products and coproducts of morphisms. Hence, distributive laws of products and coproducts can be constructed and studied. Because diamonds are based on the interplay of categories and saltatories, which are involved with two fundamental operations: composition (o) and jump-operation (||), it is reasonable to find interactive laws as distributivity between those basic operators inside the very definition of the conception of diamonds.

Similar distributivity of products and coproducts can then be introduced, not only for categories but for saltatories, too. And diamond products and coproducts with their internal and external distributivity can be studied.



Pawel Sobocinski, 2007
http://www.mimuw.edu.pl/~tarlecki/teaching/ct/slides/Warszawa1.pdf

"A category with finite products and finite coproducts is said to be distributive, if for all objects A, B, and C, the canonical map

∂ : A x B +A x C –> A x (B +C) is invertible.

These categories have proved to be important in theoretical computer science as they facilitate reasoning about programs with control and the specification of abstract data types."
(J.R.B. Cockett, Stephen Lack) http://www.tac.mta.ca/tac/volumes/8/n22/n22.pdf

Another source
http://www.mathematik.uni-marburg.de/~gumm/Papers/Distributivity.pdf

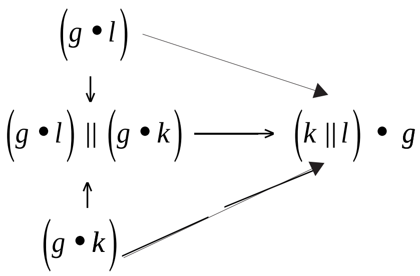
Distributivity constructions for diamonds

A diamond with composition and jump-operation (sautisition) is said to be diamond-distributive, if for all morphisms and hetero-morphisms f, g, k, l, the diamond-canonical map

d-∂: (g • l) || (g • k) —> (k || l) • g is diamond-invertible.

By analogy to the category definition of distributivity we can state:

In any diamond with composition and complementation, the following map exists, using the properties of composition and complementation:



Distributivity

$\left(k \parallel l\right) \bullet g = g \bullet \left(k \parallel l\right)$

$\left(k \parallel l\right) \bullet g = \left(g \bullet l\right) \parallel \left(g \bullet k\right)$

$\left(k \parallel l\right) \bullet g = \left(g \bullet l\right) \; o \; \left(g \bullet k\right)$

$\left(k \parallel l\right) \bullet g = \left(g \bullet l\right) \bullet \left(g \bullet k\right)$

Interactivity

$$\begin{bmatrix} \left(k \parallel l\right) \; o \; g \\ g \; o \; \left(k \parallel l\right) \end{bmatrix} = \begin{bmatrix} \left(g \parallel l\right) \; o \; \left(g \parallel k\right) \\ \left(g \; o \; l\right) \parallel \left(k \; o \; g\right) \end{bmatrix}$$

The interactional composition (combination) *(l o g o k)* can be read in different ways:

1. From the position of morphism g, there is an "arrival" hetero-morphism l and a "retro-grade" hetero-morphism k for g.

2. From the position of the hetero-morphisms k, l, there is a bridging morphism g, connecting both hetero-morphisms.

3. The bridging compositions are symmetric:
(k || l) • g = g • (k || l).

4. Distributivity can be interpreted as a form of interactivity. For diamonds, distributivity is an interactivity between categories and saltatories relized by the operations of composition and saltitions.

Interaction between categories and saltatories in diamonds

Reversion of Diamonds

$$[\mathbf{A};\ \mathbf{a}] \in Diam \ \ iff \ \ rev([\mathbf{A};\ \mathbf{a}]) \in Diam$$

$$rev([\mathbf{A};\ \mathbf{a}]) = [\mathbf{a};\ \mathbf{A}]$$

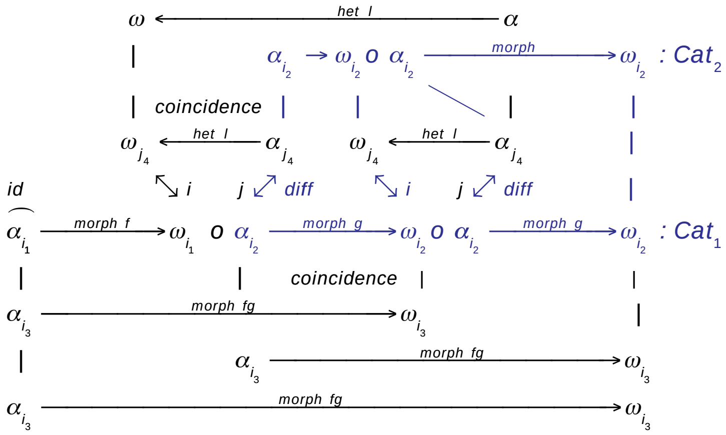
$$rev([\mathbf{A};\ \mathbf{a}]) = [compl(\mathbf{A});\ compl(\mathbf{a})]$$

$$\left.\begin{array}{l} compl(\mathbf{A}) = \mathbf{a} \\ compl(\mathbf{a}) = \mathbf{A} \end{array}\right\rangle rev([\mathbf{A};\ \mathbf{a}]) = [\mathbf{a};\ \mathbf{A}]$$

The reversion of a diamond is a diamond.

6.5.2 Modeling of n-categories in diamonds

Architectonics for a 2 - category

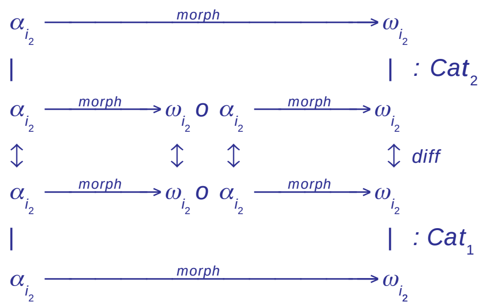


A further speculation might be risked.

Given the diamond notion of *difference* and *coincidence,* a 2-categorical construction of morphisms seems to be possible.

With this construction, the structural differences between different categories of n-categories might be structurally identified. The construction suggests that different categories are located in a differential chain of categories/saltatories/categories. Hence, in this sense, a 2-category is not simply an abstract second category, studying the relation between morphisms, but a category of a different level of abstraction and based on the difference between categories and saltatories. This is realized by a chain of *difference* and *coincidence* operations between categories and saltatories. A direct modeling with the help of the difference operation, applied to the categorical morphisms, wouldn't work because category theory is not offering such a difference operation.

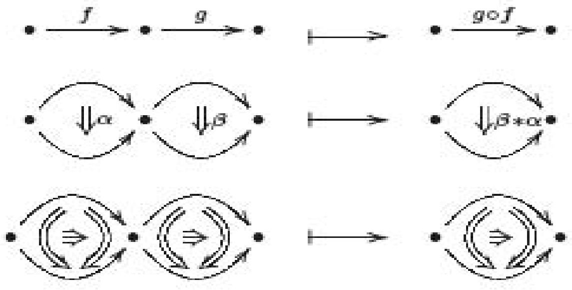
Short version for diamond-based 2-categories



As a consequence, the distribution of categories in n-categories would be motivated and constructed by established operations, here of diamond theory. It seems, that the common introduction of n-categories as generalizations of 1-categories has some ad-hoc characteristics which are not motivated by intrinsic operations of general category theory or universal algebra. An attempt to fill this conceptual gap is proposed by the idea of *categorification* (Baez).

This short version of the diagram is "forgetting" the different steps of the construction of the 2-category.

n-category diagrams]

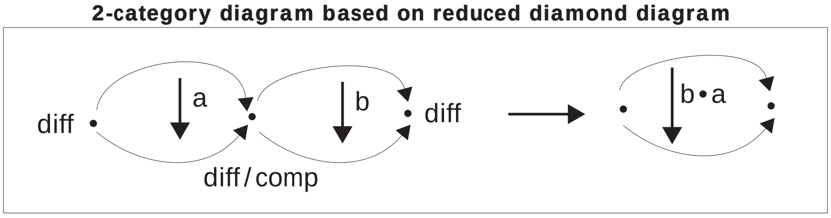


The first diagram is representing 1-categorical composition.

The second diagram is representing 2-categorical composition, i.e., composition beween morphisms of two categories.

The following diagram shows a modeling of a 2-category by a reduced diamond diagram.

2-category diagram based on reduced diamond diagram



Differences are enabling matching conditions which are not based on coincidence alone. Thus, they are offering more general meeting points for morphisms or cells than it is offered by the matching conditions based on coincidence of domain and codomain.

The main idea of this construction, again, is to give the "distributed" categories of an n-category a structural definition of their position in the "field" of distributed categories. It seems that n-category theory is still blind for the question of *positionality*.

In n-category, a quite "helpless" anchoring of the different categories is given by the highly obscure presupposition of the coincidence of "domains" and "codomains" for all categories at a common point. This is proposed as a meta-theoretical statement for all categories, and not produced "inside" of the different categories of an n-category. Hence it is a decision, not made conceptually intelligible, based on the fear to loose ground. This argument is not loosing its point with the introduction of more complex topologies or knotted figures.

The diamond modelling is not only giving some insight into the mist of this obscurity and the conceptual pointlessness of the point-anchors, but is also enabling categories to "run" in parallel, not restricted by "globularity", which is disallowing parallelism, or not. Globularity as a primary concept is pointless. It is useful only as a secondary and reductional concept of a chiastic mediation of different categories.

Within other metaphorics, we have to zoom into the points of globularity to discover the intriguing chiastic structure of the matching conditions, suppressed by ordinary n-category theory.

Some definitions from the experts

"4.1. *Basic data.* First we recall the definition of *globular* sets. The idea is that we have, for each natural number n, a set X(n) of "n-cells", each of which has a source (n – 1)-cell and a target (n – 1)-cell. The source and target of a cell must themselves share the same source and target; this is the "globularity" condition, ensuring that the cells have "globular" shape rather than any other kind of shape.

Formally, the category of globular sets **GSet** has objects X consisting of sets X_n for each natural number n and maps s_n, t_n : X_n –> X_{n-1} subject to the relations ss = st, tt = ts; morphisms f : X –> Y in **GSet** consist of maps f_n : X_n –> Y_n that strictly commute with the s_n and t_n."

"*Remarks*. The condition forcing the 0-cells here to be trivial ensures that this putative n-category is globular (cells satisfying the globularity conditions ss = st, ts = tt) and not "cubical". This might seem unnatural for an n-category of cubes, but in the case of manifolds in cubes it arises because our 0-manifolds can always be embedded in the open cube $(0, 1)^m$ leaving the edges "empty". However, for more complicated TQFTs such as open-closed TQFT it may be natural and / or necessary to drop this "globular" condition and build cubical n-categories. Cubical n-categories have a similar flavour to n-categories but raise some very different issues; as for n-categories they are currently only well understood in low dimensions or strict cases." (Cheng, Gurski)

Towards An N-category Of Cobordisms, Eugenia Cheng And Nick Gurski
http: / / www.tac.mta.ca / tac / volumes / 18 / 10 / 18-10.ps.

Obviously, and again, there is nothing wrong with n-category theory as it is, it is simply not the game I would like to play.

Have fun with the Eugenia from the Catsters: http: / / www.youtube.com / TheCatsters

6.5.3 Subversionality of Diamonds

Hetero-morphisms and morphograms

Instead of leaving category theoretic terms and topics for kenogrammatics, the neither-nor-question for objects and morphism is leading to hetero-morphisms of rejectionality. This approach was not yet conceived in the study *"Categories and Contextures"*.

"Given the basic concepts of category theory we are free to apply the Diamond Strategies to re-design the field.

With the basics of objects and morphism naturally 4 positions can be focused.

First, the classic focus, is on objects. The categorial results are statements about objects in categories.

Second, the more modern focus is on morphisms. Here even objects are conceived as special morphisms. Both thematizations are of equal value especially because the terms "object" and "morphism" are dual. More interesting are the two further steps of diamondization of the categorial basics "object" and "morphism".

Third, we ask *"What is both at once, object and morphism?*" An answer is given by the distribution and mediation (dissemination) of categories in a *poly-categorial* framework.

Forth, the question arises:*"What is neither object nor morphism?"*

Also the following citation of Gunther does not intent to gives a definitional clear explanation of a *neither-nor* situation it is useful as a hint in the right direction.

„Thus the proemial relation represents a peculiar interlocking of exchange and order. If we write it down as a formal expression it should have the following form:

$$\square \; R^{pr} \; \square$$

where the two empty squares represent kenograms which can either be filled in such a way that the value occupancy represents a symmetrical exchange relation or in a way that the relation assumes the character of an order.“ Gunther, p. 227

Obviously, the scheme or formula, represents neither an order nor an exchange relation. With this in mind, we can try to think the *neither-nor* of objects and morphisms of category theory as the inscription of the processuality of „categorization“ in itself into a scriptural domain beyond classical formal systems, that is into *kenogrammatics*.

We need this quite wild „anti-concept“ of kenogram and kenogrammatics to deal scientifically and technically with the structure of any change, the proemiality, which is not to catch by any construction based on semiotical identity." p. 7 (Kaehr)

http://www.thinkartlab.com/pkl/lola/Categories-Contextures.pdf

Despite an obvious kind of similarity of the complementary pair "morphisms" and "hetero-morphisms" in diamond theory in respect of the terms "object" and "morphisms", it seems to be reasonable to understand hetero-morphisms as belonging to a realm which is governed neither by categorical objects nor categorical morphisms. Hetero-morphisms don't belong to categories but to "*saltatories*" which are studying the "morphisms" of the realm of rejectionality. Categories are studying the morphisms of the field of acceptionality. Both, categories and saltatories together, are inscribing the interplay of diamonds.

Interplay of morphisms and morphograms

"In mathematics, a *morphism* is an *abstraction* of a structure-preserving mapping between two mathematical structures.

A *category* C is given by two pieces of data: a *class* of *objects* and a class of *morphisms*.

There are two operations defined on every morphism, the *domain* (or source) and the *codomain* (or target).

For every three objects X, Y, and Z, there exists a *binary operation*
hom(X, Y) x hom(Y, Z) —> hom(X, Z) called *composition*." WiKi

The "double gesture" of inscription is not enfolded as a succession of different contextural decisions. It is given/installed at once. Hence, there is some similarity in the description of diamond objects to morphograms. Morphograms are inscribing standpoint-free complexity. But there is also another approach to morphograms.

As Heinz von Foerster proposed, morphograms can be regarded as the *inverse* function of a logical function. Hetero-morphisms are inverse to morphisms. Hence, there is a possible connection between hetero-morphisms of a composition and morphograms of such a composition. In this sense, morphograms can be seen as the inscription of the inversion of morphisms, i.e., of rejectional morphisms. But hetero-morphisms as inverse morphisms are not simply dual to morphisms, they are not only "morphisms" with an inverse arrow to acceptional morphisms, they are on a different level of *abstraction*, too. Because morphisms are mapping between objects, and hetero-morphisms are abstractions from the operator of composition, their conceptual status is principally different. Morphisms are mappings as mappings; hetero-morphisms are abstractions from the interaction of morphisms. Hence, the new couple in diamonds is: *morphism/morphogram*.

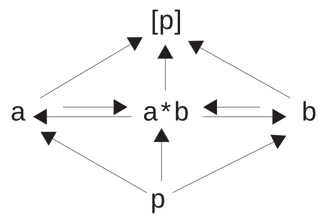
Objects in diamond systems are based on as-abstractions. The core system is abstracted by its acceptional and/or rejectional aspect. There is no neutral object in diamonds like in the lambda calculus. Reference in the lambda calculus is an identification of an object as an identity. This identity can be simple or complex (composed) but its naming and reference is realized by a simple operation of identification, establishing the identity of the object.

Thus, the fundamental properties of hetero-morphisms before questions of identity/diversity and commutativity, associativity properties are studied, are:

1. *inverse morphism* property
2. *actional abstraction* property

These two properties are defining the rejectional status and the saltatory structure of jumpoids.

An accessible, and first interpretation of the two properties of hetero-morphisms can be found in the theory of morphogrammatics. Morphograms can be regarded as inversa of compositions. They are "object-free, thus, more abstract than morphisms. But as morphograms of compositions they are connected to compositions of morphisms. They may be seen as generalizations of compositions of abstract morphisms.



The categorical product "a*b" is founded in p. The categorical product is based on the inverse product, the thematization of the compositor, as a morphogram [p]. The core elements of the diagram, a, b, a*b, have a double meaning. They belong to categories and to saltatories. Insofar, they define the structure of the morphogram [p].

As an example, we can think of a logical disjunction "a v b", which is based on its constituents "a" and "b" as core elements. These together can be inverted to the hetero-morphism [p], which defines the morphogram of the binary disjunction as the operativity of the operator "v", but concretized in its complication, as a binary action, by the constituents "a" and "b".

Because morphograms can be conceived as inversa of compositions, and are generating a generalization of the composition of morphisms, they are representing a permutation-invariant class of compositions. In the example, the morphogram [p] is representing the disjunction "avb" as well as all negations of it "¬(avb)". Hence, again, morphograms are negation-invariant patterns.

If a product composition is called a *process* (Baez) then the complement of the process is the form or structure of the process, hence inscribed as the morphogram of the process.

6.5.4 Positionality of Diamonds

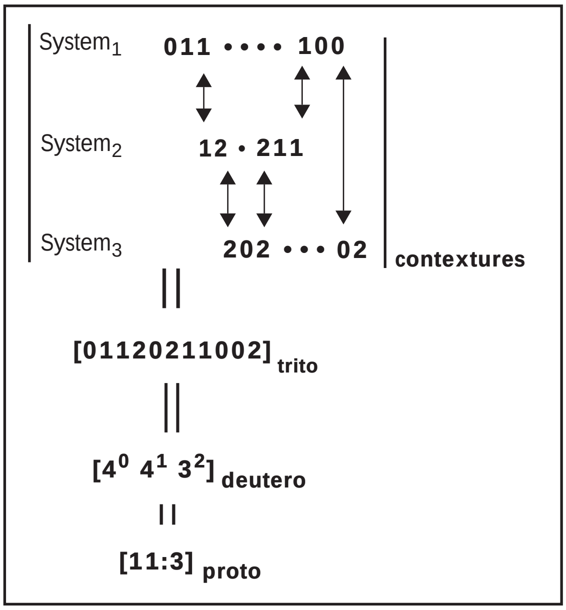
Levels of situatedness of diamonds

1. Diamonds in proto-mode: Distribution of Diamonds onto the proto-structure,
2. Diamonds in deutero-mode: Distribution of Diamonds onto the deutero-structure,
3. Diamonds in trito-mode: Distribution of Diamonds onto the trito-structure,
4. Diamonds in logic-mode: Distribution of Diamonds onto polylogical-structure,
5. Diamond-structure of the modi of distribution [proto, deutero, trito, logic].

Diamonds are directly produced by the operations of iteration and accretion in proto- and deutero-structures and their commutativity. The case is more intricate for trito-structures. The proposed solution is locating, at first, diamonds inside of trito-grams and not between trito-grams of different complexity as for proto- and deutero-grams. Thus it is introducing iteration and accretion inside of the trito-gram and not between trito-grams of different complexity. More correctly, the path producing the tritogram can be interpreted in different ways, thus enabling commutativity. To discover a commutativity between different trito-grams for trito-arithmetic iteration and accretion is another question.

Abstractions

The aim of this endeavour is to develop a mechanism to give the diamonds a concrete position, a *structural place*, before / beyond classical logical systems. Such a placement of diamonds can be succeed on different levels of pre-logical structures, i.e., the kenogrammatic structures of proto-, deutero- and trito-differentiation. Beyond logic, i.e., beyond mono-contexturality, a distribution of diamonds in poly-contextural situations is proposed. The diamond strategies, short the diamonds, are explanations of the metaphor of tetraktomai, i.e., of doing the tetraktys, and its translation into the strategy of diamondization.



Abstractions and concretizations between the levels may help to gain a better understanding of the strategy.

open / closed world assumption

Epistemologically, diamonds are conceived as the structure of a closed world in a cosmology of an open multitude of worlds. Hence, diamonds are the diamonds of a multitude of worlds and have to be localized by their world. Diamonds are per se disseminated, i.e., distributed and mediated, and not in isolation.

Only for introductional purposes, diamonds are presented in this study mainly in isolation.

7 Axiomatizations of Diamonds

7.1 Axiomatics One

$$\mathbf{Diamond}^{(m)}_{\mathbf{Category}} = \left(\mathbf{Cat}^{(m)}_{\mathbf{coinc}} \mid \mathbf{Cat}^{(m-1)}_{\mathbf{jump}}\right)$$

$$\mathbb{C} = (M, o, \|)$$

1. Matching Conditions

a. $g\ o\ f,\ h\ o\ g,\ k\ o\ g$ and

$b_1 \xleftarrow{l} b_2$

$c_1 \xleftarrow{m} c_2$

$d_1 \xleftarrow{n} d_2$

$l \parallel m \parallel n$ are defined,

b. $h\ o\ ((g\ o\ f)o\ k)$ and

$b_1 \xleftarrow{l} b_2 \parallel c_1 \xleftarrow{m} c_2 \parallel d_1 \xleftarrow{n} d_2$

$l \parallel (m \parallel n)$ are defined

c. $((h\ o\ g)\ o\ f)o\ k$ and

$(l \parallel m) \parallel n$ are defined,

d. mixed : f, l, m

$l \parallel m, \bar{l}\ o\ f\ o\ \overline{m}$

$(\bar{l}\ o\ f)\ o\ \overline{m}$,

$\bar{l}\ o\ (f\ o\ \overline{m})$ are defined.

2. Associativity Condition

a. If $f, g, h \in MC$, then $h\ o\ ((g\ o\ f)\ o\ k) = ((h\ o\ g)\ o\ f)\ o\ k$ and

$l, m, n \in MC$ $\quad l \parallel (m \parallel n) = (l \parallel m) \parallel n$

b. If $\bar{l}, f, \overline{m} \in MC$, then $(\bar{l}\ o\ f)\ o\ \overline{m} = \bar{l}\ o\ (f\ o\ \overline{m})$

3. **Unit Existence Condition**

a. $\forall f\ \exists (u_C, u_D) \in (M, o, \|) : \begin{cases} u_C\ o\ f,\ u_D\ o\ f, \\ u_C \parallel f,\ u_D \parallel f \end{cases}$ are defined.

4. Smallness Condition

$\forall (u_1, u_2) \in (M, o, \|) : hom(u_1, u_2) \wedge het(u_1, u_2) =$

$$\left\{\begin{matrix} f \in M \mid f\ o\ u_1 \wedge u_2\ o\ f, \\ f \in M \mid f \parallel u_1 \wedge u_2 \parallel f \text{ are defined} \end{matrix}\right\} \in SET$$

8 Complexity reduction by diamondization

8.1 Reduction steps

category —> duality of category ==> complementarity of duality of category.

Hence, in diamond theory, Herrlich's principle *"two for the price of one"* holds too. But because of the diamond abstraction, which is reducing complexity, we have less to carry home.

Diamond theory is dealing with duality for categories and for saltatories and with the complementarity between saltatories and categories and their dualities.

$$diamonds = \begin{bmatrix} saltatory \\ category \end{bmatrix} x \begin{bmatrix} duality \\ complementarity \end{bmatrix}$$

$$\begin{array}{ccc} Cat & \overset{compl}{\rightleftarrows} & Salt \\ \updownarrow dual & & \updownarrow \\ Dual & \overset{compl}{\rightleftarrows} & Dual \end{array}$$

$$\forall f_1 ... f_m : length\left(f^{(m)}\right) = \binom{m}{2}$$

$$\forall u_1 ... u_n : length\left(u^{(n)}\right) = \binom{m-1}{2}$$

$$comp\left(f_1, ..., f_m\right) = f^{(m)}$$

$$comp\left(u_1, ..., u_n\right) = u^{(n)}$$

$$length\left(f^{(m)}\right) > length\left(u^{(n)}\right)$$

8.2 Reduction by morphograms

Hetero-morphisms as morphograms are enabling a further reduction of complexity.

Levels of Diamond Abstractions

$$X = g \,\square\, f = \left[\left(g \,\lozenge\, f\right);\left[u\right]\right] = \left[\left(g \ o \ f\right);\left(u\right);\left[u\right]\right]$$

Category *: level of composition :* $\left(g \ o \ f\right)$

Saltatory *: level of complementarity :* $\left(u\right)$

Morphogrammatics *: level of subvertionality :* $\left[u\right]$

Reduction steps

$$length\left(g \,\square\, f\right) < length\left(g \,\lozenge\, f\right) < length\left(g \ o \ f\right)$$

8.3 Diamonds as complementations of Categories

It could be said that diamond theory is simply a complementation aspect of categories. It may be a new, perhaps strange operation on categories but based nevertheless on categories and therefore not a concept in its own right. There are many operations possible on categories, especially duality, why not complementarity? With that, we could stay firm to category theory and, if it makes any sense at all, add the operation of complementation to its main operators.

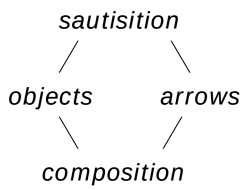
In this sense, diamond theory would have the merit of introducing an new operation to the known categorical operations – and nothing more. It may even be the case, that the diamond operations and notions are appearing somewhere in category theory in a different form not yet accessible to my understanding.

Hence, we would have *compl*: [Cat] —> [Cat; diam].

Against this reduction procedure I would like to argue that it is missing the point.

The only similar operation in category theory to complementation seems to be dualization. But dualization is not part of the very definition of categories but is a meta-theoretical property of categories while diamonds, i.e., saltatories, which are complementary to categories are introduced on the very basic level of the definition of diamonds. Duality is a meta-theoretical concept, complementarity an object-language or proto-theoretical concept and strategy. Even if diamondization is regarded as a meta-theoretical concept, its concepts and strategies have to be defined on a proto-language level. There seems to be no reasonable arguments to introduce complementation as a meta-theoretical concept like it happens for dualization. Dualization is natural as a reversion of arrows and is naturally motivated by the basic concepts of category: arrows and objects. But there is no natural motivation to introduce complementation for categories. Diamonds are realized in a different paradigm of thinking than categories.

Categorical Diamond



In other words, category theory is based on a duality of objects and morphisms (arrows), diamond theory is based on the genuine 4-fold structure of diamonds, i.e., class of objects, class of arrows, neither-nor of objects and arrows, the collection of hetero-morphism and the both-and of objects and arrows, the collection of compositions of morphisms.

That is, (objects, arrows, composition) belong to Class-1 while (object, arrows, sautisition) belongs to Class-2. Class-1 is the class of morphisms. Class-2 is the class of hetero-morphisms. Class-1 and Class-2 are mediated in bi-conglomerates.

Another candidate to reduce diamonds to categories could be seen by the index- or *fibre-categories*. Fibering was used to formalize polycontextural logic by Jochen Pfalzgraf. This is of help to deal mathematically with polycontextural systems. But here again, it has to be mentioned, that the diversity and multitude of contextures in polycontexturality is introduced at the very proto-logical level of the formalism while the strategies of fibering are secondary and are based on mono-contextural category theory. Hence, fibering and similar concepts are not doing the job their purpose is different.

As a consequence of the proto-logical status of the diamond definition, meta-theoretical techniques like dualization and fibering can be applied, secondarily, on diamonds as such too.

Duality and complementarity; again

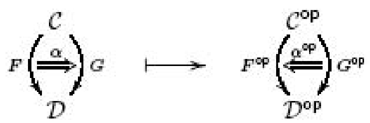
It has clearly to be distinguished between the complementary definition of categories and saltatories in diamonds from the complementarity *operations* in diamonds, like *acc* and *rej*, which are transforming category (saltatory) formulas into saltatory (category) formulas. The complementarity operation can be considered as a meta-theoretical operation with a similar status to the dualization operation in categories. Dualization can be applied to saltatories and to diamonds, too.

In category theory there are no *bridging* rules between dual categories. Bridging rules are a mix of the sides of the mirror of diamond complementarity: categorical as well as saltatorical. The category of dual categories would have to be introduced to study categorical duality as an interplaying concept.

> "More generally a statement S involving a category C automatically gives a dual statement S^{op} obtained by reversing all the arrows. This is known as the duality principle." (C´accamo)

Duality Revisited

The operation taking a category $\mathcal{C}$ into $\mathcal{C}^{op}$ can be extended to an endofunctor over **CAT**. Indeed $\_^{op}$ defines actions for objects, functors and natural transformations given in reality a 2-functor. It acts over categories as follows:



where $\alpha_c^{op}: G^{op}(c) \to F^{op}(c)$ in $\mathcal{D}^{op}$ corresponds to the arrow $\alpha_c: F(c) \to G(c)$ in $\mathcal{C}$. Then this functor is contravariant with respect to natural transformations only.

By definition of opposite category for objects $c, d \in \mathcal{C}$ there is an isomorphism of sets

$$\mathcal{D}(F(c), G(d)) \cong \mathcal{D}^{op}(G^{op}(d), F^{op}(c)). \tag{1.1}$$

http://www.brics.dk/DS/03/7/BRICS-DS-03-7.pdf

Isomorphism of duality

Is there a similar isomorphism for complementarity as for duality?

The mapping for the duality principle is strictly symmetric, i.e., the number of morphisms and composition operators is equal for the category and its duality.

The mapping between categories and saltatories for complementarity is not symmetrical, i.e., the mapping is not one-to-one. A categorical triple [morph1, morph2, comp] is mapped onto the saltatorical one-tuple [het]. Or in other words, a dyadic operation "comp" is mapped onto an ununary operation, i.e., a single morphism "het". Additional to the numeric asymmetry a conceptual asymmetry occurs: *operation* (composition) of category to an *operand* (hetero-mophism) of saltatory.

The question is, how to construct the categorical situation out of the saltatorical?