Toth's semiotic diamonds

Analyzing construction principles for semiotic diamonds

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Abstract

A a detailed comparison of Toth's semiotic diamonds (Diamanten) and the diamonds of diamond category theory is presented. It turns out that Toth's Diamanten are based on inversions of acceptional morphisms and are not constituting any rejectional morphisms. A proper definition of the matching conditions is missing. A comparison of the matching conditions for Diamanten and diamonds gives easy criteria for separation of the approaches. As a result, semiotic Diamanten are not working as semiotic models of categorical diamonds. Nevertheless, semiotic Diamanten are a novelty in semiotics and are opening up new fields of semiotic studies.

Sketch of Toth's semiotics

The semiotic composition operation of the elementary semiotic mappings, like (I→M), (M→O), (I→O), between the objects I, M, O, is *commutative* and *associative*. And obviously the *identity* mapping *id* is realized for the objects I, M, O.
Hence, semiotic composition can be studied as a mathematical category in the sense of category theory with objects I, M, O and its mappings (arrows) between the objects.

Categorical interpretation of the semiotic sign scheme

$$\begin{aligned}
&i.i = \text{id}i,\ i = 1, 2, 3 \\
&1.2 = \alpha \equiv 1 \xrightarrow{\alpha} 2 \\
&1.3 = \beta\alpha \equiv 1 \xrightarrow{\beta\alpha} 3 \\
&2.3 = \beta \equiv 2 \xrightarrow{\beta} 3 \\
&2.1 = \alpha^{\circ} \equiv 1 \xleftarrow{\alpha^{\circ}} 2 \\
&3.1 = \alpha^{\circ}\beta^{\circ} \equiv 1 \xleftarrow{\alpha^{\circ}\beta^{\circ}} 3 \\
&3.2 = \beta^{\circ} \equiv 2 \xleftarrow{\beta^{\circ}} 3
\end{aligned}$$

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3,\ 1 \xrightarrow{\beta\alpha} 3$$

$$1 \xleftarrow{\alpha^{\circ}} 2 \xleftarrow{\beta^{\circ}} 3,\ 1 \xleftarrow{\alpha^{\circ}\beta^{\circ}} 3$$

In concreto, it still has to be analyzed how the semiotic *matching conditions* for compositions are realized.
In the example above, the question is, how is "2.1" in (3.1 → 2.1) as a *codomain* and in (2.1 → 1.3) as a *domain* defined?

The new question which arises for abstract diamond theory is: How are the difference relations and hence the hetero-morphisms defined *in concreto*?

I had the idea that the papers of Toth are filling this gap with his *semiotic* modeling of diamonds. But it turned out that there is a cricial difference between ‘semiotische Diamanten” and categorical diamonds.
Therefore, there is still an open question about a semiotic concretization of categorical diamonds.

Toth is *suggesting* an answer to the question, how to interpret the *difference* relations, with the introduction of the operation of *inversion* INV of a concrete sign scheme. Hence, Toth’s interpretation of diamonds tries to join together semiotic and diamond thematizations and notational systems.

Inversion

Where is this operation INV from?
The sign relation ZR is defined as a relation of *monadic*, *dyadic* and *triadic* relations:

$ZR = (a, (a \Longrightarrow b), (a \Longrightarrow b \Longrightarrow c))$.

Sign values for ZR are:
a = {1.1, 1.2, 1.3}
b = {2.1, 2.2, 2.3}
c = {3.1, 3.2, 3.3}

ZR = <3.x, 2.y, 1.z> with $x,y,z \in \{1,2,3\}$ and $x \leq y \leq z$.

It is clear, that the semiotic inversion operation INV is a semiotic operation based on the elementary operations of transpositions and is not leading out of the semiotic domain.
INV is defined by:

INV(a.b c.d e.f) = (e.f c.d a.b).

In contrast, dualization is defined as:

DUAL(a.b c.d e.f) = (f.e d.c b.a).

The *abstract sign scheme* gets an interpretation by the introduction of the instances:
I = interpretant, M=medium and O=object.

Hence, the semiotic triad occurs as morphisms between the instances I, O, M and their combinations, called graph theoretic sign models.

1. (I⟶O ⟶M)	4. (O⟶M⟶I)
2. (M⟶O⟶I)	5. (I⟶M⟶O), (M⟶I⟶O)
3. (I⟶M⟶ O)	6. (O⟶I⟶M).

It is proposed by the Stuttgarter School of semiotics (Bense, Walter) that those triadic sign schemes can be composed by their dyadic relations (mappings).
Most of the semiotic work is in German, thus it is easily possible that I will miss the correct terminology.

Example:

2. $(M \Longrightarrow O)\ (O \Longrightarrow I) = (M \Longrightarrow O \bullet O \Longrightarrow I)$
Hence, triadic-trichotomic sign relations are compositions of *dyadic*-dichotomic relations. This is a strong thesis, and I don’t see the necessity of such a reduction.
Even more problematic, Elisabeth Walter (1979, S. 79), speaks of a *lattice theoretical* union of “$(M \Longrightarrow O)\ (O \Longrightarrow I) = (M \Longrightarrow O.O \Longrightarrow I)$". (Toth, (2008b), p.11)

Bense (1976) is mentioning a category theoretical composition of the triadic sign conceived as a transition from the set theoretic and relational definition to a more adequate presentation (Darstellung) of semioticity .

In the following, I will first follow this strategy, then I will focus on the composition of triadic-trichotomic sign structures as such.
The diamondization of the internal relations of signs might be called *micro*-analysis, the focus on the latter *macro*-analysis of semiotic diamonds.

These 6 graph theoretic sign models of I, O, M, get an interpretation by their corresponding numeric value occupancies.
Example
3.1 (I⟶O⟶M)
(3.1 2.1 1.1)   (3.1 2.3 1.3)
(3.1 2.1 1.2)   (3.2 2.2 1.2)
(3.1 2.1 1.3)   (3.2 2.2 1.3)
(3.1 2.2 1.2)   (3.2 2.3 1.3)
(3.1 2.2 1.3)   (3.3 2.3 1.3)
(Thot, p. 2, 2008a)

The new question which arises now is: How are the *difference* relations of the semiotic diamond and hence the hetero-morphisms defined *in concreto*? More precisely, how are the difference relations between the domains/codomains of morphisms and hetero-morphisms of semiotic composition defined? And is the inversion INV operation strong enough to define the differentness of the new hetero-morphisms?

Response to Toth’s remark

My question: *“Is the inversion INV operation strong enough to define the differentness of the new hetero-morphisms compared to morphisms?”*

Toth’s remark
"In einer kürzlich veröffentlichten Kritik bemerkte Rudolf Kaehr zurecht, dass in dergestalt eingeführten semiotischen Diamanten die Heteromorphismen nichts anderes seien als *Spiegelungen* dyadischer semiotischer Funktionen (Kaehr 2008, S. 3).

Kaehr übersieht allerdings, dass die *Umkehrungen* dyadischer Funktionen nur *formal*, aber nicht *inhaltlich* Spiegelungen sind. Z.B. bedeutet (2.1 ⇒ 3.1) die rhematische Interpretation eines Abbildes, aber die umgekehrte Funktion (3.1 ⇒ 2.1) muss, wie bereits Bense (1981, S. 124 ff.) bemerkte, nicht zum selben Icon zurückführen. Es kann sich hier also um einen echten semiotischen *Heteromorphismus* handeln.” [my emph]

Response
As far as I understand, Toth is pointing to the fact, that an inversion (Umkehrung, Spiegelung) of a morphism like (2.1 ⇒ 3.1) is not re-establishing the identity of “2.1” *in concreto*.
In other words, the formal *inversion* of a dyadic function is not identical with its mirroring with regard to its *content*. Hence, with an inversion INV (2.1 ⇒ 3.1) = (3.1 ⇒ 2.1), the role of “2.1” might change from a rhematic interpretation of an object to a different interpretation of an object, not necessary the same icon.

The same is true for the inversion of morphisms in a category! But, nevertheless, category theory is studying morphisms and their inversions and duality *in abstracto*. The same is intended for diamond theory. Morphisms and heteromorphisms in diamonds are considered as *abstract*.

That’s a reason too, while Toth’s approach to diamonds is important: he offers a concrete interpretation. Hence, it also could be understood as a *model* for abstract diamonds. On the other hand, diamonds are instrumental to solve some intriguing problems of mathematical semiotics.

With my introduction of diamonds and their hetero-morphisms I tried to distinguish them from both, inversions and duals (opposites) of morphisms. It is well known, that an inverted, also dual, morphism is still a morphism for which all the properties and laws for morphisms (identity, commutativity, associativity) of the category holds. While for hetero-morphisms, different laws are involved. And that's the reason why they are called *hetero-morphism* and are belonging to *saltatories* and not to categories.

What I called *antidromic* direction of arrows for hetero-morphisms is not simply an *inversion* of the arrow of a morphism, and therefore still a morphism, but a new abstraction based on "*difference*" relations of the "targets" and "sources" of composed morphisms, i.e. of *compositions*. Hetero-morphisms are conceived as abstractions from the operation of *composition* and not from morphisms between objects. Even if those difference relations are considered as inversions, they have to take place at the right place, i.e. at the target of the first and at the domain of the second morphism of the composition of the morphisms.

This in full harmony to what I developed from the very beginning of the introduction of the diamond concept (diamond category theory, diamond theory, etc.).

The name *"hetero-morphisms*" might be misleading, but the formal definition is what counts and not its label. There is a different use of the term 'hetero-morphism' in my paper *"Categories and Contextures".*

Because diamonds are introduced in an *abstract* way, i.e. depending on the *alpha/omega-structure* of composition of morphisms only, it is of importance to find reasonable examples as *concretizations*. An other interesting concretization is the attempt to introduce diamond *relations* as a diamondization of relation theory.

In this sense of *concretization*, I see the importance of Toth's approach, albeit he is missing the train.

Unfortunately, after I finally understood Toth's concept "semiotischer Diamanten" (semiotic diamonds) it turned out that Toth's "*Diamanten*" are strictly different from my 'diamonds'.

Hence, Toth has given some interesting interpretations for his "Diamanten" in the context of semiotics but not to my diamonds. That is, my hope for a semiotic concretization of the abstract mathematical concept of 'diamond category theory' has still to wait to be achieved.

Nevertheless, there are also *creative* misunderstandings! In this sense, both concepts, the "Diamanten" and the diamonds, are interesting topics. Toth's "Diamanten" are, thus, a (creative) misinterpretation of my diamonds.

Hence, despite the examples for 'semiotische Diamanten', the relevant question still is: How are *semiotic* diamonds defined? Where are the properties, rules and laws? And, first of all, how are the *difference* relations introduced? There is surely an inversion operation in semiotics but there is no abstraction corresponding to the diamond difference relations or operations.

Semiotics is still depending on is-abstractions in contrast to as-abstractions in diamond theory.

Diamonds, everywhere?

Independent of the question of the motivation to mention that my teacher Gotthard Gunther had introduced diamonds long before my own humble trial, Toth's move confirms clearly and without sophisticated manoeuvres of my interpretations what exactly he understands by a "Diamant" (diamond). Correctly, Toth understands by a Diamant the formal structure of a special square: ◇. Unfortunately, not all such diamonds are diamonds in the sense of the *diamond category theory* I introduced recently. What counts in a mathematical theory are the definitions, properties and rules. The same holds for diamond (category) theory.

I'm quite convinced that my definitions, despite their tentativeness, are clear enough to show the difference between a '*square*' concept and my diamond concept of an interplay between *categorical* and *saltatorical* arrows (maps, morphisms).

In other words, diamonds, labeled with some mighty labels might seduce to engage into a recreational game but are by no way of any interest for my own work, which stands without any 'didactical' figures.

The other hint to Toth's diamond interpretation is given by his *historical* comments that I introduced myself the diamond approach first 1995 and then specially in 2007. I appreciate the honor I' m given for a figure, which is known, at least outside polycontexturality, since tausends of years.

In fact, I'm turning around in this carousel at least, officially, since 1973-75 (published 1978), with my *closed* proemial relationship ("geschlossene proemial Relation"), which Gunther liked very much, albeit I didn't see much to formalize.

It was writen strategically against the Varela's ECI and the re-entry mysticism.

A university degree was achieved by my student Klaus Grochowiak with *"Die formal Struktur der Zirkulation bei Karl Marx"*, 1976, which is an intriguing application of diamonds, albeit labelled differently, and chiasms!

Then, the very first diamond approach for communicational purposes (and money making) was developed 1992 and was called, for a joke, *"Das Existenz-Halma"*. Halma is a board game. It worked! Why not? And again, I started a series of serious work about *"Diamond Strategien"* (1995-1998).

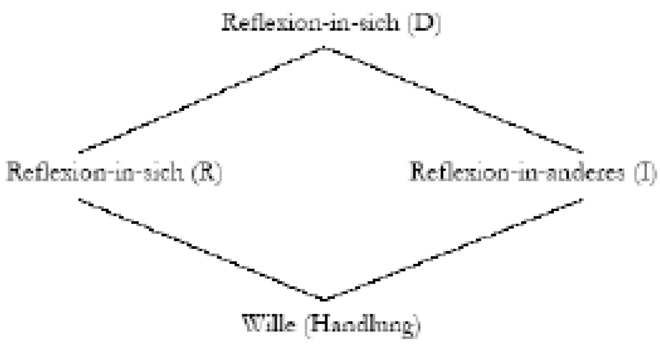
Without doubt, I'm very much aware about that, i.e. my own intellectual history. Too long to tell! It would be surprisingly strange or even highly cranky if I would have proclaimed **2007** that I discovered the diamond of category theory or diamond category theory, not knowing that I and especially Gunther introduced diamonds long before.

Again, the whole story is documented in my "*Handbuch*" (1995)! I simply would like to recall Daniel Hellerstein's *Diamond Logic*! Which nevertheless has nothing to do with diamond theory.

As often, misunderstandings might create some interesting results.
But scholasticism of some groups is forcing me to offer some clarifications.

Nevertheless, it is a merit of Alfred Toth to have done interesting work on the base of his concept of Diamanten and he has mentioned his source of inspiration as far as he has access to it.
On the other hand, I feel motivated to typset even more constructions.

Alfred Toth, Phantasie und Technik, 1.1.2009
"Offiziell wurde der Diamant als logisches Modell erst durch Kaehr (1995) und vor allem Kaehr (2007) in die Polykontexturalitätstheorie eingeführt. Allerdings findet man bereits in Günthers "Bewusstsein der Maschinen" einen höchst interessanten Diamanten im Zusammenhang mit der reflexionstheoretischen Begründung einer Theorie des Willens im Sinne einer Theorie transzendentaler Handlungen:



Toth's general procedure

"Die semiotische Rejektionsfunktion ist nun aber keineswegs auf den Strukturtyp (e.f c.d a.b) wie im obigen semiotischen Diamanten beschränkt. Semiotische Inversion (INV) ist allgemein durch folgende zwei Anweisungsschritte erreichbar:

1. Kehre die *Reihenfolge* der konstituierenden Subzeichen einer Zeichenklasse (oder einer ihrer Transpositionen bzw. Dualisationen) um.
2. Vertausche alle semiotischen Morphismen mit ihren *Inversen* (wobei natürlich z.B. a°°= a, b°° = b und per definitionem (vgl. Toth 1993, S. 21 ff.) (ba)° = a°b° und (a°b°)°= ba gilt."

INV – Rules
I. $a^{\circ\circ} = a$, $b^{\circ\circ} = b$,
II. $(ba)^{\circ} = a^{\circ}b^{\circ}$,
III. $id^{\circ} = id$
$(a^{\circ}b^{\circ})^{\circ} = (b^{\circ\circ}a^{\circ\circ}) = ba$ (IV.)

Hence, there are two steps to consider for the construction of a hetero-morphism in a semiotic diamond:
1. ***Chiasm****: Change the order of subsigns.*
2. ***Inversion****: Exchange all semiotic morphisms with their inversion.*

This can be reformulated by:
1. Apply the *inverse* operation INV to the parts (A_3, B_3) of an (acceptional) morphism ($morph_3$).
An *acceptional* morphism is the composition of two basic morphism of a semiotic category.
2. *Substitute* the results of the *first* part, (A_3), of the morphism with the *second* part, (B_4), of the hetero-morphism, $morph_4$, to be constructed. And the same procedure with the *second* part, (B_3); substitute it with the *first* part (A_4) of the hetero-morphism $morph_4$.
And additionally:
3. ***Positioning, difference, matching conditions***
3.1 The second part of the first morphism, its codomain or target, and the first part of the hetero-morphism, its domain, source, have to be matched.
3.2 The first part of the second morphism, its domain or source, and the second part of the hetro-morphism, its codomain, target, have to be matched.

A hetero-morphism is the rejectional morphism of an acceptional morphism. But this distinction has first to be established. Without the distinction between acceptional and rejectional morphism, the constructed hetero-morphism is not yet placed. To realize its correct placement, a new condition has to be fulfilled. It is the role of the *difference* relations to organize such a placement of a not yet positioned hetero-morphism.
It seems that Toth's approach is not considering this part of the construction.

Semiotic hetero – morphism construction
$\forall$ morph (A_3, B_3), (A_4, B_4) $\in$ SEMIOTICS
IF
(A_3, B_3) $\in$ Morph
$Subst_{(INV(A_3)/B_4)} \wedge Subst_{(INV(B_3)/A_4)}$
THEN
(A_4, B_4) $\in$ Het – Morph.

Schemes of inversion and exchange

***a)* intended scheme?**

$$\boxed{\overline{C} \longleftarrow \overline{A}} : \text{heter}$$

| ? | ?

$$(A \longrightarrow B)\, o\, (B \longrightarrow C) : \text{comp}$$

| |

$$\boxed{A \longrightarrow C} : \text{morph}$$

***b)* realised scheme!**

$$\boxed{\overline{C} \longleftarrow \overline{A}} : \text{heter}$$

| |

$$(A \longrightarrow B)\, o\, (B \longrightarrow C) : \text{comp}$$

| |

$$\boxed{A \longrightarrow C} : \text{morph}$$

Example

$$A_3 = [\alpha^\circ \beta^\circ, \alpha] \longrightarrow [\text{id}1, \beta] = B_3$$

$$\downarrow \text{ INV, Subst } \downarrow$$

$$B_4 = [\beta\alpha, \alpha^\circ] \longleftarrow [\text{id}1, \beta^\circ] = A_4$$

$$\overleftarrow{[[\text{id}1, \beta^\circ], [\beta\alpha, \alpha^\circ]]} = \text{heter}_4$$

$$\mathbf{x}$$

$$\underrightarrow{[[\alpha^\circ \beta^\circ, \alpha], [\text{id}1, \beta]]} = \text{morph}_3$$

Toth’s examples are showing a solution in the sense of scheme (a), i.e. a hetero-morphism (heter_4) is defined as a direct inversion of an acceptional morphism (morph_3). And morph_3 is a composition of morph_1 and morph_2.

Inversion (categorical dualisation)
It seems that the first step of ‘chiastic exchange’ is included in the definition of the inversion operation.

Example

1. Chiastic exchange (" external exchange ")

$A_3 = [\alpha^\circ \beta^\circ, \alpha] \longrightarrow [\text{id}1, \beta] = B_3$

$\text{INV}(A_3 \longrightarrow B_3) = \text{INV}(B_3) \longrightarrow \text{INV}(A_3)$. This is the ' exchange' (by INV – RuleI – II)

$= \text{INV}(A_3) \longleftarrow \text{INV}(B_3)$

2. Inversions (" internal exchange ")

$A_3 = [\alpha^\circ \beta^\circ, \alpha] = ([\alpha^\circ \beta] \longrightarrow \alpha)$

$B_3 = [\text{id}1, \beta] = (\text{id}1 \longrightarrow \beta)$

$\text{INV}(A_3) = \text{INV}([\alpha^\circ \beta] \longrightarrow \alpha)$:

$= (\text{INV}(\alpha) \longrightarrow (\text{INV}([\alpha^\circ \beta^\circ]))$: INV – RuleI – II

$= (\alpha^\circ \longrightarrow [\alpha\beta]) = A_4$

$\text{INV}(B_3) = \text{INV}(\text{id}1 \longrightarrow \beta)$

$= (\text{INV}(\beta) \longrightarrow \text{INV}(\text{id}1)$: INV – RuleI – II

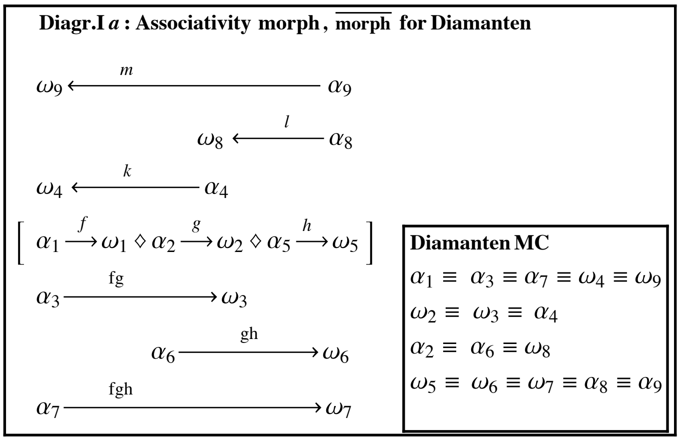
$= (\beta^\circ \longrightarrow \text{id}1) = B_4$ •

Toth’s notation for brackets [x, y] in [x, y], [z]] of a hetero-morphism could mislead to a wrong interpretation if taken as a mapping "(x)⟶ (y), but [x, y] is by definition, in this setting, a non-decomposable “object".

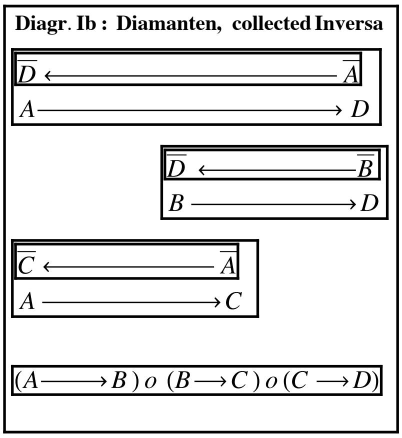
Further comparisons: Are Diamanten diamonds?
Hence, the difference to my approach is clear. Hetero-morphisms are not directly accessible by acceptional morphisms but are depending on the structure of the composition of morphisms. That is, the process of composition is reflected in hetero-morphisms, while acceptional morphisms are the positive or direct results of compositions, i.e. $morph_1 o\, morph_2 = morph_3$.

A consequence of Toth's approach seems to be the lack of jumping situation or *saltisitions* for general Diamanten. Toth's hetero-morphisms are connected and composed without any gaps to be over-jumped and hence, the associativity rule holds unrestrictedly.

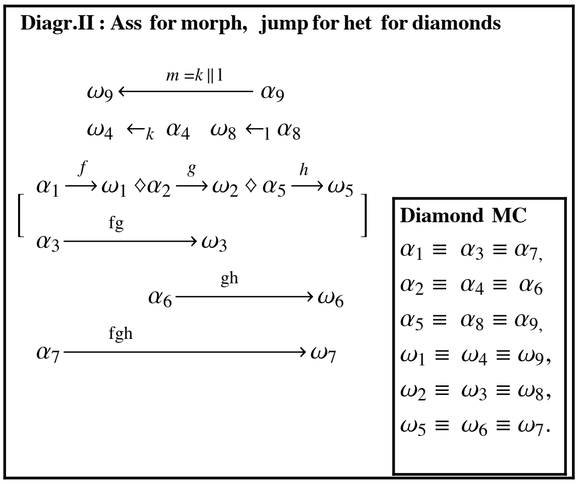
The main difficulty to understand properly Toth's approach has two reasons:
1. the diagrams used are not always clear. Different readings are possible, because
2. there are no formal *matching conditions* (MC) for compositions of morphisms and hetero-morphisms defined.



Diagr.I a : Associativity morph, $\overline{\text{morph}}$ for Diamanten

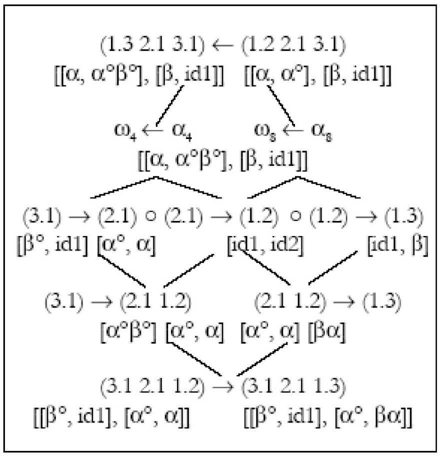


Diagr. Ib : Diamanten, collected Inversa



Again, because diagrams are not telling much without their formal definitions, I repeated the formal definitions of the diagrams for Diamanten and for diamonds.
And, unfortunately, programs are still not stable enough to produce proper results in different formats.

The above diagram is semiotically concretised by Toth’s example for a semiotic Diamant based on 3 composed semiotic morphisms.



As a first result we might observe that Toth’s hetero-morphisms are a kind of *counter-morphisms*. They might even have the properties of being: *“... genuinely tetradic, chiastic, antidromic and 4-fold.”* And they are connected with morphisms by *inversion* and *chiastic* exchange. Obviously those properties are necessary but maybe not sufficient for diamonds.

What I proposed as diamonds at different places, are structures with very different laws compared to the laws of categories. That is, diamonds, which consist of a complementary interplay of categories and saltatories, are as categorical systems identitive, commutative and associative in respect to their objects, morphisms and composition. Therefore, inheriting all the laws and methods from category theory. In sharp contrast, saltatories as parts of diamonds, are ruled by difference, jumps (saltisitions) and jump-associtivity, etc.
Additionally, diamonds as such, are containing *bridges* and *bridging* rules between categories and saltatories.

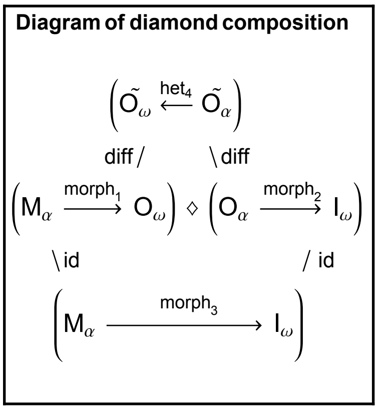
▫ *Again, diamond definitions*

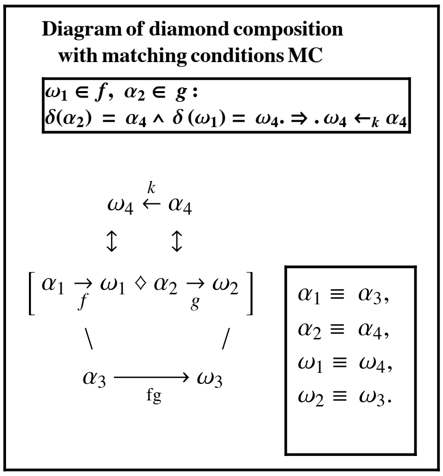
Diamond composition rule $(M_\alpha \rightarrow O_\omega) \diamond (O_\alpha \rightarrow I_\omega)$

$(M_\alpha \rightarrow I_\omega) \,\big|\, (\tilde{O}_\omega \longleftarrow \tilde{O}_\alpha)$

$\tilde{O}_\alpha \equiv \text{diff}(O_\alpha)$

$\tilde{O}_\omega \equiv \text{diff}(O_\omega)$.





One of Toth's semiotic motivations for diamonds

"Ein sowohl für die Semiotik wie für die Kybernetik wichtiges Ergebnis ist, dass zwar die Abbildung von Zeichenklassen auf Morphismen eindeutig ist, nicht aber die Abbildung von Morphismen auf Realitätsthematiken: Während also $\alpha° \longrightarrow$ (2.1), id2 $\longrightarrow$ (2.2), $\beta\longrightarrow$ (2.3) gilt, erhalten wir für ((2.1), (2.1)), ((2.2), (2.2), (2.2)) und ((2.3), (2.3)) jedesmal id2, d.h. bei den Semiosen zwischen Trichotomischen Triaden, die ja aus Realitätsthematiken konstruiert sind, können die für Zeichenklassen eingeführten Morphismen nicht zwischen den trichotomischen Stellenwerten unterscheiden!

Der Grund dafür hängt wohl mit der Tatsache zusammen, dass dynamische (generative) Semiosen und (degenerative) Retrosemiosen im Gegensatz statischen Subzeichen und Zkln/Rthn nicht umkehrbar-eindeutig sind, weshalb in Toth (2008a) der Versuch unternommen wurde, sie mit Rudolf Kaehrs Hetero-Morphismen im Rahmen seiner polykontexturalen Diamanten-Theorie zu identifizieren (vgl. Kaehr 2007).

Sollte diese Identifikation statthaft sein, kommt denjenigen Trichotomischen Triaden, die wie etwa die Nrn. 589 und 617 ausschliesslich aus semiotischen Heteromorphismen (und idx) und vor allem denjenigen, die (neben) idx nur Morphismen und ihre korrespondierenden Heteromorphismen enthalten wie etwa den Nrn. 551, 561 und 564, besondere Bedeutung im Zusammenhang mit einer erst zu entwickelnden kybernetischen Semiotik der 2. Ordnung bzw. einer Vereinigung von Semiotik und Polykontexturalitätstheorie zu." (Toth, Kybernetik, p.663/64)

589 [II, OO, MM] $\Longleftrightarrow$ [3.1 × 3.2 × 3.3 - 2.1 × 2.2 × 2.3 - 1.1 × 1.2 × 1.3]
$\Longleftrightarrow [\alpha^{\circ}\beta^{\circ}\,\beta^{\circ}$ id3 - α° id2 β - id1 $\alpha\ \beta\alpha]$

3.1 × 3.2 × 3.3	2.1 × 2.2 × 2.3	3.1 × 3.2 × 3.3
T1 : 2.1 × 2.2 × 2.3	T2 : 1.1 × 1.2 × 1.3	T3 : 1.1 × 1.2 × 1.3

$b'1 = [\beta^{\circ}, \beta^{\circ}, \beta^{\circ}]$ $b'2 := [\alpha^{\circ}, \alpha^{\circ}, \alpha^{\circ}]$ $b'3 = [\alpha^{\circ}\beta^{\circ}, \alpha^{\circ}\beta^{\circ}, \alpha^{\circ}\beta^{\circ}]$
$\bigcap b' i = \emptyset$ (p.595)

Playing the game

Now I would like to test Toth's construction rules (INV, exchange) in reconstructing his examples.
It seems, that the INV rule is always working if applied in the introduced sense as "INV = inv(inv, inv)" . With the help of the exchange advise, INV is reduced to "(inv, inv)".

Again, Toth's example and an explicite notation:

$[\alpha, \alpha^{\circ}\beta^{\circ}] \leftarrow [\beta, \mathrm{id}_1]$: morph$_4$
/ \
$[\beta^{\circ}, \mathrm{id}_1] \diamond [\alpha^{\circ}, \beta\alpha]$
\ /
$[\beta^{\circ}, \mathrm{id}_1] \longrightarrow [\alpha^{\circ}, \beta\alpha]$: morph$_3$

explicite notation

$[\alpha \leftarrow (\alpha^{\circ}\beta^{\circ})] \leftarrow [\beta \leftarrow \mathrm{id}_1]$: morph$_4$
/ \
$[\beta^{\circ} \rightarrow \mathrm{id}_1] \diamond [\alpha^{\circ} \rightarrow \beta\alpha]$: morph$_1$ *o* morph$_2$
\ /
$[\beta^{\circ} \rightarrow \mathrm{id}_1] \longrightarrow [\alpha^{\circ} \rightarrow (\beta\alpha)]$: morph$_3$

- ***Example1***

(1.3 × 1.2 × 3.1)
$[[\mathrm{id1}, \beta^{\circ}], [\beta\alpha, \alpha^{\circ}]]$

$[\alpha^{\circ}\beta^{\circ}, \alpha]$ o $[\mathrm{id1}, \beta]$

$[[\alpha^{\circ}\beta^{\circ}, \alpha], [\mathrm{id1}, \beta]]$
(3.1 × 1.2 × 1.3)

$$1.\ A_3 = [\alpha^\circ\,\beta^\circ, \alpha] \Longrightarrow A_4 = [\beta\alpha, \alpha^\circ] = [\beta\alpha \leftarrow \alpha^\circ]$$
$$\downarrow \qquad\qquad \downarrow$$
$$2.\ B_3 = [id1, \beta] \Longrightarrow B_4 = [id1, \beta^\circ] = [id1 \leftarrow \beta^\circ]$$

INVERSION INV

ad1. $INV([\alpha^\circ\,\beta^\circ, \alpha]) = INV([INV([\alpha^\circ\,\beta^\circ]), INV([\alpha])$
$= [INV([\alpha]), INV([\alpha^\circ\,\beta^\circ])$
$= [[\alpha^\circ], [\beta\alpha]] = [\alpha^\circ, \beta\alpha] = [\alpha^\circ \longrightarrow \beta\alpha] = \mathbf{A_4}$

ad2. $INV([id1, \beta]) = INV([INV(id1), INV(\beta)])$
$= [INV(\beta), INV(id1)]$
$= [\beta^\circ, id1] = [\beta^\circ \longrightarrow id1] = \mathbf{B_4}$

- ***Example2***

$$(3.1 \times 2.1 \times 1.3)$$
$$[[\beta^\circ, id1], [\alpha^\circ, \beta\alpha]]$$
$$[\alpha, \alpha^\circ\,\beta^\circ] \quad o \quad [\beta, id1]$$
$$[[\alpha, \alpha^\circ\,\beta^\circ], [\beta, id1]]$$
$$(1.3 \times 2.1 \times 3.1)$$

$$1.\ A_3 = [\alpha, \alpha^\circ\,\beta^\circ] \Longrightarrow A_4 = [\alpha^\circ, \beta\alpha] = [\alpha^\circ \leftarrow \beta\alpha]$$
$$\downarrow \qquad\qquad \downarrow$$
$$2.\ B_3 = [\beta, id1] \Longrightarrow B_4 = [\beta^\circ, id1] = [\beta^\circ \leftarrow id1]$$

INVERSION INV

ad1. $INV[\alpha, \alpha^\circ\,\beta^\circ] = [\alpha^\circ, \beta\alpha]$
$= INV(INV[\alpha], INV[\alpha^\circ\,\beta^\circ])$
$= INV[\alpha^\circ\,\beta^\circ], INV[\alpha]$
$= [[\beta\alpha], \alpha^\circ] = [\beta\alpha, \alpha^\circ] = [\beta\alpha \longrightarrow \alpha^\circ] = \mathbf{A_4}$

ad2. $INV[\beta, id1] = [\beta^\circ, id1]$
$= INV[INV(\beta) \to INV(id1)]$
$= [INV(id1) \to INV(\beta)]$
$= [id1 \to \beta^\circ] = \mathbf{B_4}$

Are Toth's hetero-morphism and semiotic diamonds the same constructs or constructs in the same spirit as the diamond categories introduced by my own intuitions? What could the difference be? And how could such a possible difference matter?

Toth's construction is considering the complementarity between acceptional and rejectional morphisms based on inversion (INV) and chiastic exchange.
This corresponds to some sketches I produced myself. But I conceived them as *abbreviations* of the difference based constructions.

"Compositions as well as sautisitions (jump-operations) are ruled by identity and associativity laws. Complementarity between categories and saltatories, i.e., between acceptional and rejectional domains of diamonds, are ruled by difference operations." (Kaehr, p.3)

▫ *Comparision: Toth's Diamanten and diamonds*

The following definitions could give a hint to understand the *difference* between the two diamond constructions.

Complementarity of Acc and Rej based on diff

$X \in$ Acc iff compl(X) $\in$ Rej

$X = g \; o \; f$:

1. $X \in$ Acc if compl(X) $\in Rej$

compl($g \; o \; f$) = compl (compl(g) o compl(f))

= compl(diff(cod(f)) o diff(dom(g)))

= compl$\left(\left(\overline{B_{cod}}\right) o \left(\overline{B_{dom}}\right)\right) = \omega_4 \leftarrow \alpha_4$.

(u : $\omega_4 \leftarrow \alpha_4$) $\in$ Rej

Hence, ($g \; o \; f$) $\in$ Acc if (u : $\omega_4 \leftarrow \alpha_4$) $\in$ Rej

($g \; o \; f$) $\in$ Acc if $\left(g \; \overline{o} \; f\right) \in$ Rej.

2. compl(X) $\in$ Rej if $X \in$ Acc

compl($\omega_4 \leftarrow \alpha_4$) = compl(compl($\omega_4$) $\leftarrow$ compl(α_4))

= compl(($A_{dom} \rightarrow B_{cod}$) $\leftarrow$ ($B_{dom} \rightarrow C_{cod}$))

= (($A_{dom} \rightarrow B_{cod}$) o ($B_{dom} \rightarrow C_{cod}$))

= ($f \; o \; g$).

3. Hence, $X \in$ Acc iff compl(X) $\in$ Rej.

In this "proof", the complementarity operation "compl" is used quite freely to do also the transitional job of completing the morphisms out of the objects. This is done by the operation of "difference" and "completition", which is completing domains and codomains to their morphisms. This points to the asymmetry of Acc- and Rej-domains.

Toth's complementarity operation between acceptional and rejectional morphisms, played by INV and exchange, is symmetrical.

The open question for the construction of semiotic diamonds still is: How is the difference relation defined, *concretely*?

Again, Toth's approach, despite its own merits, is not answering this questions, simply because they don't exist for his Diamanten.

▫ *Similarity?*

The nearest comparision between Toth's approach to diamond and what I published myself can be found in the general complementarity between acceptional and rejectional morphisms as sketched below. Nevertheless, the definitions of the hetero-morphisms k, l, m are based on the abstractions of compositions and not on objects. That is, hetero-morphism k : $\delta(\alpha_2) = \alpha_4$ and $\delta(\omega_1) = \omega_4$.ETC!

$\omega_1 \in f,\ \alpha_2 \in g:$

$\delta(\alpha_2) = \alpha_4 \wedge \delta(\omega_1) = \omega_4. \Rightarrow . \omega_4 \leftarrow_k \alpha_4$

$$\omega_9 \xleftarrow{m} \alpha_9$$

$$\omega_4 \leftarrow_k \alpha_4 \quad \omega_8 \leftarrow_l \alpha_8$$

$$\left[\begin{array}{l} \alpha_1 \xrightarrow{f} \omega_1 \Updownarrow \alpha_2 \xrightarrow{g} \omega_2 \Updownarrow \alpha_5 \xrightarrow{h} \omega_5 \\ \alpha_3 \xrightarrow{\text{fg}} \omega_3 \end{array}\right]$$

$$\alpha_6 \xrightarrow{\text{gh}} \omega_6$$

$$\alpha_7 \xrightarrow{\text{fgh}} \omega_7$$

Diamond MC

$\alpha_1 \equiv \alpha_3 \equiv \alpha_7,$
$\alpha_2 \equiv \alpha_4 \equiv \alpha_6$
$\alpha_5 \equiv \alpha_8 \equiv \alpha_9,$
$\omega_1 \equiv \omega_4 \equiv \omega_9,$
$\omega_2 \equiv \omega_3 \equiv \omega_8,$
$\omega_5 \equiv \omega_6 \equiv \omega_7.$

complementarity of accept, reject

reject(fg) = *k* iff accept(*k*) = (fg)
reject(gh) = *l* iff accept(*l*) = (gh)
reject(fgh) = *m* iff accept(*m*) = (fgh)

Thus, the operation *reject(gf)* of the composition of the acceptance morphisms f and g is producing the *rejectance* hetero-morphism k.
And the operation *accept(k)* of the rejectance morphism k is producing the *acceptance* of the composition of the morphisms g and f.

Comparision of Diamanten and diamonds

Diamond $MC^{(3)}$

$\alpha_1 \equiv \alpha_3,$
$\alpha_2 \equiv \alpha_4,$
$\omega_1 \equiv \omega_4,$
$\omega_2 \equiv \omega_3.$

Diamant $MC^{(3)}$

$\alpha_1 \equiv \alpha_3 \equiv \omega_4$
$\omega_2 \equiv \omega_3 \equiv \alpha_4$

plus MC for compositions

Diamond $MC^{(4)}$

$$\alpha_1 \equiv \alpha_3 \equiv \alpha_{7,}$$
$$\alpha_2 \equiv \alpha_4 \equiv \alpha_6$$
$$\alpha_5 \equiv \alpha_8 \equiv \alpha_{9,}$$
$$\omega_1 \equiv \omega_4 \equiv \omega_9,$$
$$\omega_2 \equiv \omega_3 \equiv \omega_8,$$
$$\omega_5 \equiv \omega_6 \equiv \omega_7.$$

Diamant $MC^{(4)}$

$$\alpha_1 \equiv \alpha_3 \equiv \alpha_7 \equiv \omega_4 \equiv \omega_9$$
$$\omega_2 \equiv \omega_3 \equiv \alpha_4$$
$$\alpha_2 \equiv \alpha_6 \equiv \omega_8$$
$$\omega_5 \equiv \omega_6 \equiv \omega_7 \equiv \alpha_8 \equiv \alpha_9$$

That is,

$$\forall \, \mathbf{MC}: \quad \mathbf{MC}_{\mathbf{diamond}} \neq \mathbf{MC}_{\mathbf{Diamant}} \Longrightarrow \textbf{Diamanten} \neq \textbf{diamonds}$$

This detailed comparison of Toth’s semiotic diamonds (Diamanten) and the diamonds of diamond category theory has shown some results:
- Toth’s Diamanten are based on inversions of acceptional morphisms and are not constituting any rejectional morphisms.
- A proper definition of the matching conditions is missing.
- A comparison of the matching conditions for Diamanten and diamonds gives easy criteria for separation of both approaches.
- As a result, semiotic Diamanten are not working as semiotic models of categorical diamonds.
- Nevertheless, semiotic Diamanten are a novelty in semiotics and are opening up new fields of semiotic studies.