Generalized Diamonds

From monosemic to tectonic complementarity

Rudolf Kaehr Dr.[@]

ThinkArt Lab Glasgow

Abstract

The construction of diamonds can be generalized towards *polysemic* and *metamorphic* interactions between categories and saltatories.

1. Generalized Diamond Conditions

1.1. Architectonics of diamonds

Composition in diamonds can be generalized towards *polysemic* and *metamorphic* interactions between categories and saltatories.

After having developed some insights and experiences with the diamond approach and its complementary structures, a design of diamond category theory might be introduced which is not as close to the introductory analogy to classic category theory. Following the classic strategy of academic research a *generalization* of the introduced concepts of diamond category theory shall be sketched.

To some degree, such generalizations are obvious, but nevertheless quiet intriguing albeit a tedious pleasure.

Asymmetry in the interplay

The first introduction of the diamond category concept is based on the strict and primary distinction of categorical objects and morphisms and their composition. A saltatorical hetero-morphism is thus an abstraction from the composition operation on morphisms resulting in an asymmetry between categories and saltatories. A composition is defined on 2 morphism, an abstraction on the composition is establishing a single hetero-morphism as a reflection of the categorical composition activity in a saltatory. Hence, a commutative composition of 2 morphisms is mirrored by only one hetero-morphism. Thus, the commutativity of the composition of 2 morphism has no direct proper correspondence in the commutativity of a single heteromorphism.

Therefore, the general sentence *"To each commutativity in a category a commutativity in a saltatory corresponds"* leads to conflicts if we use the strict and restricted introduction of diamonds.

Diamond Commutativity

$$(g \diamond f) = \chi \langle (g \circ f) \amalg I \rangle$$

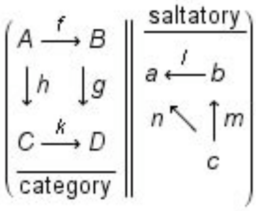
$$\text{with} \begin{pmatrix} \text{cod}(f) \cong \text{dom}(g) \\ \omega(f) \Updownarrow \alpha(g) \end{pmatrix}$$

$$(g \circ f) \amalg I: \text{ with commuting scheme}: \left(\frac{\begin{matrix} A \xrightarrow{f} B \\ h \searrow \ \downarrow g \\ C \end{matrix}}{\text{category}} \middle\| \frac{\text{saltatory}}{a \xleftarrow{I} b} \right)$$

Associativity Condition

Iff, $g,\ h,\ k \in MC$, and $l,\ m,\ n \in MC$:

$$then \begin{bmatrix} h \circ ((g \circ f) \circ k) = ((h \circ g) \circ f) \circ k \\ l \amalg (m \amalg n) = (l \amalg m) \amalg n \end{bmatrix} \text{ such that}$$



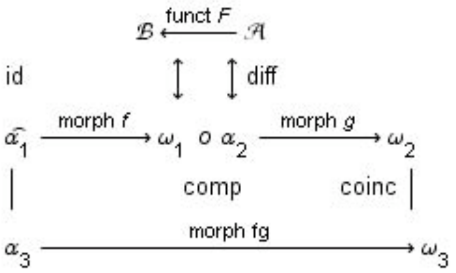
commutes in category and salataory.

Balancing the interplay

A. Complementarity of features (properties, structures, data):
1. in parallel, commutativity to commutativity
2. mixed, say commutativity to associativity
B. Bridging features of complementarity

All those combinations are possible with a liberalization in the definition of the constitutive rules for diamonds.
That is, the complementarity of a categorical composition has not to be represented in a single elementary hetero-morphism it could be mapped into a complexion of hetero-morphisms.
With that, a free, but still reasonable mixture of features could be realized.

Architectonics



$$\mathcal{A} = \left(\omega_4 \xleftarrow{\text{het } l} \alpha_4\right),\ \mathcal{B} = \left(\omega_4 \xleftarrow{\text{het } l} \alpha_4\right)$$

This example could be read as a complementary distribution of a categorical composition of morphisms, hence a category, and a saltatorial functorial mapping of saltatories. Hence, the difference operation is not reduced to polysemy but is a mapping between morphisms of a category and functors in a saltatory. Such a mapping is crossing tectonic levels, here, between morphisms and functors.

Functors are mappings between categories, thus, in our case, they are *hetero-functors* as mappings between saltatories. Functors in categories are associative under composition, in saltatories they are associative under saltisition. That is, the jump-operation holds not only for hetero-morphisms but for hetero-functors too.

For alone standing categories it seems not to make any sense to mix morphisms with functors in one design. For diamonds, the possibility to mix types between categories and saltatories is opening up a new kind of flexibility in modeling complex systems.

Standard diamond definitions

bi – Object $[X, x]$

$$\left.\begin{array}{l} \widehat{id} \\ x \in Salt \\ \updownarrow diff \\ X \in Cat \\ \widehat{id} \end{array}\right\} \in Diam$$

Identity

$\forall f, X, Y, o \in Cat:$

$f \, o_{XXY} \, ID_X = f = ID_Y \, o_{XYY} \, f.$

$\forall l, x, y, \shortparallel \in Salt:$

$l \shortparallel_{xxy} ID_x = l = ID_y \shortparallel_{xyy} l.$

Difference

$OmCat, Salt \in Diam:$

$\forall [X, x], [Y, y] \in Diam$

$[f, l] (o \shortparallel)_{[XYy, xyY]} DIFF_{[Y, y]}$

$= [f, l] =$

$DIFF_{[y, Y]} (\shortparallel, o)_{[xyY, XYy]} [l, f].$

Identity is a mapping onto-itself as itself.

For each object X of a category an identity morphism, ID$[X, X]$, which has domain X in the category and codomain X in the same category exists. Called ID$_X$ or id$_X$ for ID$[X, X]$.

For each object x of a saltatory an identity morphism, ID$[x, x]$, which has domain x in the saltatory and codomain x in the same saltatory exists. Called ID$_x$ or id$_x$ for ID$[x, x]$.

Difference is a mapping onto-itself as other.

For each object X of a category a difference morphism, DIFF$[X, x]$, which has domain X in the category and codomain x in the saltatory exists. For each object x of a saltatory a difference morphism, DIFF $[x, X]$, which has domain x in the saltatory and codomain X in the category exists.

For each cat-object X an identity ID$_X$ in Cat(X, X). For each salt-object x an identity ID$_x$ in Salt(x, x) exists. And, for each bi-object [X, x] a difference DIFF$[X, x]$ between Salt(x, x) and Cat(X, X).

Tectonics of Diamonds

According to the presentation of categories by Eugenia Chang, a category consists of Data, Structure and Properties (DSP). Categories as graphs with structure are defined as DSP in the following sense:

Definition: A **category** is given by

i) **Data**: a diagram $\left(C_1 \underset{t}{\overset{s}{\rightarrow}} C_0\right)$ in *Set*

ii) **Structure**: composition and identities

iii) **Properties**: unit and associativity

A *first* step in developing a tectonics for diamonds is introduced by an *inversion* of the full DSP-scheme (Data, Structure, Properties) from DSP to PSD. That is, the properties are determining the choice for the structure and data of the structuration.

A *second* step is diamondizing PSD.

- Diamonds are conceived as an *interplay* of categories and saltatories, hence PSD has to be distributed and involved into a complementary and chiastic interplay, resulting in: YPSD.

- Disseminated Diamonds, YPSD, are involved into interactionality and reflectionality as iterative and accretive *interactions*, resulting in IYPSD.

- Interacting YDSPs are localized and positioned into the kenomic grid by the *place-designator*, resulting in LIYPSD.

Hence the diamondized DSP results into the LIY(PSD)- archictecture.

DSPYIL-Architectonics of diamonds

i) **Data**: 2-diagram C_1-s,t-->C_o/C_o<-diff-C_1 in 2-Set,

ii) **Structure**: composition, identities + saltistition, difference,

iii) **Properties**: unit, associativity + diversity, jump law,
iv) **Interplay**: complementarity and chiasm between category and saltatory,
(v) **Interactions**: diamonds with diamonds, iterative/accretive,
vi) **Localisation**: kenomic grid, place-designator.

A *third* step is freely interchanging the structure and property features of categories and saltatories in the sense of metamorphic transformations realized by the super-operators.
Free mixtures of structures (commutativity of composition and identities) with properties (unit, associativity) and its saltatorical equivalents (saltisition, difference) shall be introduced.

$$\text{Interplay}(\text{Diam}) = \chi((\text{Cat}, \text{ Salt}), (D, S, P))$$

$$\chi((\text{Cat}, \text{ Salt}), (D, S, P)) = \begin{pmatrix} DD & DS & DP \\ SD & SS & SP \\ PD & PS & PP \end{pmatrix},$$

$$\text{with } (XY) \equiv \text{Cat}(X) \longrightarrow \text{Salt}(Y), \ X, Y \in \{D, S, P\}.$$

$$\text{num}(D, S, P) = 3, \ \text{num}(\chi(D, S, P)) = 3 \text{x} 3 = 9$$

1.2. Polysemic complementarity

Up to now, a *standard* interpretation was leading the construction of diamond category theory. That is, the range of the *difference* relation as part of the definition of the bi-object of diamonds placed between categories and saltatories had to be monosemic and preserving the tectonics of the categories, i.e. objects to objects and morphisms to morphisms. That is, between a categorical object X and a saltatorical object x, a 1-1-mapping was supposed.
Polysemic-Mappings:

one – to – one
$$\textbf{bi – Object } [X, x]$$
$$\left.\begin{array}{l} \widehat{\text{id}} \\ x \in \text{Salt} \\ \updownarrow \text{diff}_{(1-1)} \\ X \in \text{Cat} \\ \widehat{\text{id}} \end{array}\right\} \in \text{Diam}$$

one – to – many
$$\textbf{bi – Object } [X, x^{(m)}]$$
$$\left.\begin{array}{l} \widehat{\text{id}} \\ x^{(m)} \in \text{Salt} \\ \updownarrow \text{diff}_{(1-m)} \\ X \in \text{Cat} \\ \widehat{\text{id}} \end{array}\right\} \in \text{Diam}$$

many – to – one
$$\textbf{bi – Object } [X^{(m)}, x]$$
$$\left.\begin{array}{l} \widehat{\text{id}} \\ x \in \text{Salt} \\ \updownarrow \text{diff}_{(m-1)} \\ X^{(m)} \in \text{Cat} \\ \widehat{\text{id}} \end{array}\right\} \in \text{Diam}$$

many – to – many
$$\textbf{bi – Object } [X^{(m)}, x^{(n)}]$$
$$\left.\begin{array}{l} \widehat{\text{id}} \\ x^{(n)} \in \text{Salt} \\ \updownarrow \text{diff}_{(m-n)} \\ X^{(m)} \in \text{Cat} \\ \widehat{\text{id}} \end{array}\right\} \in \text{Diam}$$

This decision for a mono-semic approach is guaranteeing the diamonds a strong stability. But it also can be regarded as a restriction. Hence, a polysemic and trans-tectonic approach shall be introduced.

Polysemic relations in regard of the basic terms of identity and difference shall be sketched.

Protological comment

From a *proto-theoretical* point of view, some comments about the status of the difference relation would be appropriate. The usual problem of *use* and *mention* of terms, here "relation", in a case of *abuse* of terms, is demanding for justification. If the concept of relation is entirely covered by categories and the difference between categories and saltatories is alien to categories, how has the concept of a relationship between categories and saltatories be deconstructed to model both, its status as a proper relation and as concept of relationship beyond its proper definition of relation? This question remains in the to-do box.

1.3. Tectonic metamorphosis

Minimal tectonics for categories is given by the 3-tupel (morphisms, functors, natural transformations).

Tectonic inter-relations between categories and saltatories:
From composition of *morphisms* to a mirroring in *hetero-morphisms,*
From composition of *morphisms* to a mirroring in hetero-*functors,*
From composition of *functors* to a mirroring in saltatorical *natural transformations.*

Metamorphism was introduced in ConTeXtures as a chiastic interplay between topics (types) of programming and contextures.

General scheme for tectonic metamorphosis:

Type – Chiasm

$$\text{Chiasm}(\text{Diam}) = \chi\left(\text{Cat}, \text{Salt}, \text{type}_{\text{Cat}}, \text{type}_{\text{Salt}}\right), \text{type} = \{\text{object}, \text{morphism}, \text{functor}, \text{natural transformation}\}$$

$$\begin{array}{ccc}
\textbf{bi-Object}\,[\mathbf{X}, \mathbf{Salt}] & \textbf{bi-Object}\,[\mathbf{Cat}, \boldsymbol{x}] & \textbf{bi-Object}\,[\mathbf{Cat}, \mathbf{Salt}] \\
\left.\begin{array}{l} \widehat{\text{id}} \\ \text{Salt} \in \text{Salt} \\ \updownarrow \text{diff}(X-\text{Salt}) \\ \text{X} \in \text{Cat} \\ \widehat{\text{id}} \end{array}\right\} \in \text{Diam} &
\left.\begin{array}{l} \widehat{\text{id}} \\ x \in \text{Salt} \\ \updownarrow \text{diff}(\text{Cat}-x) \\ \text{Cat} \in \text{Cat} \\ \widehat{\text{id}} \end{array}\right\} \in \text{Diam} &
\left.\begin{array}{l} \widehat{\text{id}} \\ \text{Salt} \in \text{Salt} \\ \updownarrow \text{diff}(\text{Cat}-\text{Salt}) \\ \text{Cat} \in \text{Cat} \\ \widehat{\text{id}} \end{array}\right\} \in \text{Diam} \\
\hline
\text{X-to-Salt} & \text{Cat-to-}x & \text{Cat-to-Salt}
\end{array}$$

Metamorphic chiasms between categories and saltatories in diamonds are supported by the generalized *difference* operation between categories and saltatories.

Polytopic Chiasms in ConTexTures

Polytopics, as a distribution of different topics over different contextures, in a reflectional and/or interactional mode, had been first introduced by the new paradigm of contextural programming, ConTeXtures. This introduction is restricted to polycontextural constellations only. The diamond approach to contextural programming wasn't yet at hand.

The following example shows a distribution of the topics *num*, *list* and *Boolean* over 3 mediated reflectional contextures of the polycontextural matrix.

samba⁽³⁾(repl, ∅, ∅) – horizon

```
[ thematize (zero, nil, false)
  [ [ identify contexture1.1
      (define zero
        (lambda ( f )
          ( lambda ( x )
              x ))) ]                                   Ø2.1   Ø3.1
    [ identify contexture1.2
      (define zero1.1 as nil
        (lambda ( l )
              true )) ]                  Ø2.2   Ø3.2
    [ identify contexture1.3
      (define nil1.2 as false
        (lambda (a b )
              b )) ]                     Ø2.3   Ø3.3  ] ]
```

ConTeXtures are dealing with types as topics, mono- and poly-topics of complex constellations of programming languages.
This *reflectional metamorphic transformation* example shows a polytopic situation with the topics *Number*,

List and *Boolean*.
Thus, "define name" is an abbreviation of "define name_i *as* name_j" with i=j, which is an application of the *as-abstraction*.
- replication *repl*, in this example, is a metamorphic replication and is not replicating isolated configurations.

Exchange relations:
- "define zero" is "define zero as zero", as the start of the electional levels. It could itself be produced by a predecessor level,
- define zero in contexture1.1 as *zero* in contexture1.1
- "define nil" is "define zero as nil",
as: define zero from contexture1.1 as *nil* in contexture1.2
- "define false" is "define nil as false".
as: define nil from contexture1.2 as *false* in contexture1.3.

Obviously, transcontextural type transformations are not identical with intra-contextural *type derivations* in the sense of the lambda calculus. The first are crossing the borders of contextures, from types in one contextures to other types in other contextures. This can happen successively, from one contexture to another contexture, or simultaneously, from a multitude of types in one or more contextures to a multitude of different types of different contextures. The lambda derivations are monocontextural in all their derivational transformations, and are not leaving their contextures, i.e. the borders of the formal system.

Diamond Chiasm Scheme

Diamond Chiasm – Scheme

$$\left[\begin{array}{l}\text{thematize chiasm}(\text{Cat}, \text{Salt}, M, \sigma) \\ \left[\begin{array}{l}\text{contexturalize} \\ \left[\begin{array}{l}\text{identify Cat} \\ \left[\begin{array}{l}\text{define } M \\ \left[\begin{array}{l}\text{state } \sigma \\ \left[\text{elect } M_{\text{Salt}}\right]\end{array}\right]\end{array}\right]\end{array}\right]\left[\begin{array}{l}\text{identify Salt} \\ \left[\begin{array}{l}\text{define } M \\ \left[\begin{array}{l}\text{state } \sigma \\ \left[\text{elect } \sigma_{\text{Cat}}\right]\end{array}\right]\end{array}\right]\end{array}\right]\end{array}\right]\end{array}\right]$$

Type – Term – Chiasm

$$\text{Diam}^{(2,1)} = \chi\left[\text{Cat}, \text{Salt}, M, \sigma\right]$$
$$= \left[\left(\left[M \longrightarrow \sigma\right]_{\text{Cat}} \updownarrow \left[M \longrightarrow \sigma\right]_{\text{Salt}}\right), \left(\left[M_{\text{Cat}} \cong M_{\text{Salt}}\right], \left[\sigma_{\text{Cat}} \cong \sigma_{\text{Salt}}\right]\right)\right]$$

Basic relations
exchange relation between M and σ, *i.e.* $M \updownarrow \sigma$
order relation between M and σ , *i.e.* $M \longrightarrow \sigma$
coincidence relation between M_i, M_j and σ_i, σ_j, i.e. $M_i \cong M_j$ and $\sigma_i \cong \sigma_j$.

$$\text{Diam}^{(2,1)} = \chi\left[\text{Cat}, \text{Salt}, M, \sigma\right]$$
$$= \left[\begin{array}{c}\text{level}_{\text{Cat}} : \left[M \longrightarrow \sigma\right] \\ \updownarrow \quad \times \quad \updownarrow \\ \text{level}_{\text{Salt}} : \left[\sigma \longleftarrow M\right]\end{array}\right]$$

A full dissemination of the type-term chiasms is distributed over 2-dimensions: the iterative and the accretive dimensions.

$$\begin{array}{lll} \text{Chiasm}^{(m,n)} = \text{level}_{\text{Cat}} : \left[\left[M \longrightarrow \sigma\right] \updownarrow \left[M \longrightarrow \sigma\right] \updownarrow \ldots \updownarrow \left[M \longrightarrow \sigma\right]\right]^{(n)} & : \left\{\text{iteration}(n)\right. \\ \qquad \updownarrow \times \updownarrow & \\ \text{level}_{\text{Salt}} : \left[\sigma \longleftarrow M\right] & \\ \qquad \updownarrow \times \updownarrow & \\ \qquad \ldots \ldots & \left\{ : \text{accretion}(m)\right. \\ \qquad \updownarrow \times \updownarrow & \\ \text{level}_{\text{Salt}} : \left[\sigma \longleftarrow M\right]\Big]^{(m)} & \end{array}$$

Catalogue of structurations

Categories: [1, 3, 1] = (1 category, 3 morphism, 1 composition (fulfilling the matching conditions)).

$$\text{level}_{\text{Cat}} : \left[M \longrightarrow \sigma\right]^1 \circ \left[M \longrightarrow \sigma\right]^2 \Longrightarrow \left[M \longrightarrow \sigma\right]^3$$

Chiasm:[1, 2, 2, 2] = (1 chiasm, 2 order, 2 exchange and 2 coincidence relations).

$$\text{Chiasm}^{(2,1)} = \chi\left[\text{Cat}, \text{Salt}, M, \sigma\right]$$

$$= \left[\begin{array}{r} \text{level}_{\text{Cat}} : \left[M \longrightarrow \sigma\right] \\ \updownarrow \times \updownarrow \\ \text{level}_{\text{Salt}} : \left[\sigma \longleftarrow M\right] \end{array}\right]$$

Polycontextural mediation: [1, 2, 3, 4] = (1 mediation, 2 exchange, 3 order, 4 coincidence relations).

$$\text{Poly}^{(2,1)} = \chi\left[\text{Cat}, \text{Salt}, M, \sigma\right]$$

$$= \left[\begin{array}{r} \text{level}_{\text{Cat}} : \left[M \cong M \longrightarrow \sigma\right] \\ \downarrow \ \updownarrow \times \updownarrow \\ \text{level}_{\text{Salt}} : \left[\sigma \cong \sigma \longleftarrow M\right] \end{array}\right]$$

Diamond: [1, 4, 2, 2] = (1 diamond with 4 order, 2 exchange and 6 coincidence relations).

$$\text{Diam}^{(2,1)} = \chi\left[\text{Cat}, \text{Salt}, M, \sigma\right]$$

$$= \left[\begin{array}{r} \text{level}_{\text{cat}} : \left[M \cong M \longrightarrow \sigma \cong \sigma\right] \\ \downarrow \ \updownarrow \times \updownarrow \ \uparrow \\ \text{level}_{\text{salt}} : \left[\sigma \cong \sigma \longleftarrow M \cong M\right] \end{array}\right]$$

Diamond plus: [1, 8, 6, 6], plus *simil* relations (cf. ConTeXtures)

$$\text{Diam}^{+(2,1)} = \chi[\text{Cat}, \text{Salt}, M, \sigma]$$

$$= \begin{bmatrix} \text{level}_{Cat} : [M \cong M \longrightarrow \sigma \cong \sigma] \\ \downarrow \times \updownarrow \;\times\; \updownarrow \times \uparrow \\ \text{level}_{Salt} : [\sigma \cong \sigma \longleftarrow M \cong M] \end{bmatrix}$$

The wording with chiastic constructions is not simply *"types becomes terms and terms becomes types"* as in a traditional chiasm but *"a type as a term becomes a term"* on a different level and, at the same time, *"a type as type remains a type"* on the same level. Thus, *a type as a term becomes a term and as a type it remains a type*. And the same round for terms.

Thus, a type has two functionalities at once, a type *as* a type and a type *as* a term.
Therefore, this double meaning has to be distributed over different localization of the complex constellation.