Reality and Human Values in Mathematics

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Abstract: Many mathematicians and philosophers say that mathematical objects have a real existence independent of any human activities or values. But do mathematicians behave as if this were true? This paper applies techniques from linguistics and sociology to show that mathematical discourse involves a highly nuanced assignment of values to objects, which is then used in resolving references to objects; it also discusses the nature of abstraction, and shows how the appearance of reality for mathematical objects arises through the use of conventions from ordinary discourse, including narrative. Results in the paper have implications for the exposition and use of mathematics, for mathematics education, and for philosophy.

1 Introduction

This paper is an empirical study of mathematics as a social activity, focused on the construction of mathematical objects, how values get attached to them, and how those values are used. Our analytic tools include ethnomethodology, discourse analysis (in the sense of sociolinguistics), cognitive linguistics, and activity theory. We will see that much interesting information is hidden in conventions that we take for granted. In particular, we will see that the modes of introduction for mathematical objects are highly nuanced, expressing the three values of mathematical importance, mathematical difficulty, and degree of existence[1]; there are even rhetorical and dramatic effects. We will also see how the transcendence of mathematical objects is achieved through discourse practice, and how mathematical objects are grounded in experience with the everyday world.

Ordinary folk find it obvious that mathematical objects[2], like the integer 329, the group of symmetries of a cube, the field of real numbers, or a proof of the infinitude of primes, are not directly perceptible in the same way as are a tree, a car, or a cow. Nevertheless, there is a school of the philosophy of mathematics which claims that mathematical objects are just as real as trees, cars, and cows — perhaps even *more* real, because they are not subject to mundane physical problems like decay and dissolution. This school is called Platonism, or mathematical realism, or just realism; see [28] for a detailed discussion. Prominent recent adherents include Roger Penrose, who used Platonism as a basis for his approach to consciousness [31, 32], and John Perry and Jon Barwise, who used realism to support their claims for the applicability of their situation theory [2]. Among numerous other Platonists are Descartes and Kant. Platonism has also been much attacked[3], and even ridiculed.

It is interesting that recent scientific research on the bases of perception, and more generally of consciousness, lends a kind of support to Platonism, by showing that our preceptions of trees, cars, cows, etc. result from complex neural processing that seems effortless to us only because it is largely unconscious; e.g., see [14] for a recent survey of consciousness studies. However, this processing is heavily conditioned by complex social factors. For example, the world does not have an inherent concept of "tree" that is distinct from that of "bush"; it could easily have happened that our concept

[1] This may sound strange, but we will see that it comes up naturally, for example, in proofs by contradiction.

[2] We take objects to include assertions and proofs, as well a numbers, sets, functions, etc.; see Section 2.

[3] Heidegger gives a short but cogent philosophical history of Platonism in Chapter 24 of [19], arguing that Plato was not a Platonist and that Nietzsche was the first to fully overcome Platonism, an insight that may have precipitated his final madness. Another notable critic is Ludwig Wittgenstein. An excellent recent attack by George Lakoff and Rafael Núñez [22] is based on cognitive linguistics.

of “tree” was different from what it is, either including some things currently excluded, or *vice versa*, or both. Any real concept is sustained and modified through social interaction, and is subject not only to problematic cases (such as bonsai), but also to different uses in different sub-communities, as well as to misunderstandings misuses, and evolution, e.g., though metaphorical extension. Moreover, the social category into which a perception is placed also effects the perception, as is brought out, for example, in James Gibson's theory of affordances [10]. Although mathematical objects might differ less from real world objects than was once thought, and future research might bring them even closer together, we can still say that Platonic objects do not appear in the real world, and that it is not possible to observe any causal role for them in any real mathematical discourse; in that sense, they do not exist.

Perhaps we can better appreciate some of the difficulties faced by a sociology of mathematics by contemplating the following two paradoxes:

1. Actual instances of mathematics are embodied, situated, and material; but mathematical objects appear to be objective and transcendental.
2. Mathematics is very abstract; but it is also very applicable.

Regarding the first, it is clear that mathematical objects are not real in the ordinary sense of tangibility; yet doing mathematics feels like working with real things. How does this happen? The second paradox has seemed particularly acute to physicists, who have produced a good deal of discussion, starting from a famous paper by Eugene Wigner [42]. We will see how these paradoxes can be resolved through certain observations about the language of mathematics.

In particular, we will examine the following questions: How are mathematical objects introduced into mathematical discourse? How are mathematical objects referenced within mathematical discourse? What role do values play in mathematical discourse (if any)? We will see that mathematical discourse builds on conventions of ordinary discourse types, especially narrative. We will look for deviations from expected usage and ask what extra *work* is being done by them. And of course, we will use what we discover to see what we can learn about the nature of mathematics. One motivation for this paper is to improve the teaching of mathematics, a subject about which there is much controversy, especially for secondary schools. Another motivation is to help design better user interfaces for mechanical theorem proving systems, since our research group at UCSD is building such a system [16, 13, 12].

Section 2 describes the data and methods that are used, Section 3 discusses related literature, Sections 4 and 5 consider how mathematical objects and assertions are introduced, Section 6 considers how they are referenced, and the nature of abstraction, Section 7 describes the discourse structure of proofs and narratives, Section 8 applies some ideas from cognitive linguistics, and Section 9 gives some conclusions and a summary. I thank Leigh Star for some valuable comments, Todd O'Brien for help with some linguistic issues, and Charlotte Linde for educating me in socio-linguistics during our long collaboration.

2 Data and Method

Our data are taken from mathematics textbooks and papers, from field notes, and from video tapes of live mathematics at a black (or white) board. We collected examples from over 20 books and papers, which happened to be in my office at the time of writing, over five hours of videotape collected in the United Kingdom and the US, all in English, and field notes from various lectures and other events. We will use the term **natural mathematics** for instances of the actual situated practice of mathematics, in textbooks, papers, or live interaction, and the term **text** for any segment taken from any one of these. We note that natural mathematics is always materially mediated [5]., e.g., by printed or handwritten symbols, or by speech (which is vibrating air).

Mathematics is a natural social activity, done by and for human beings in particular social contexts, and thus we view the language of mathematics a particular variety of natural language among many others, including narrative, plans and jokes. This approach is quite different from attempts to "purify" the language of mathematics, rendering it formal and without meaning, as in logical studies of mathematics, or rendering it purely mechanical, as in many efforts in computer science (Donald McKenzie [27] has made a careful social study of mechanized mathematics). In particular, we are interested in the values that are implicit in mathematical discourse, and that get attached to mathematical objects.

Our analysis draws on traditions that include cognitive linguistics, discourse analysis (in the sense of socio-linguistics), semiotics, and ethnomethodology. It should not be thought that we accept everything from each tradition, nor that our way of combining them is random. If we discard a certain amount of narrowness and dogmatism, not only are these traditions largely compatible, but they are also mutually reinforcing in many respects, for example, in rejecting psychological reductionism, cognitivism, Cartesian dualism, naive Platonism, and in supporting a pragmatic empiricism.

We find ethnomethodology [9, 35] valuable for its refined sensitivity to the details of interaction, its avoidance of reductionism, its strong commitment to empiricism, and its notions of "member's competance" and "accountability"; its unwillingness to use insights from linguistics, cognitive science, etc. is a drawback that is easily overcome. We take "mathematical objects" to be defined by their being used as such in mathematical discourse; that is, we take our warrant for this term from the way that discourse participants speak or write. This does not imply that we (or they) are necessarily commited to a philosophical position which affirms (or denies) that mathematical objects actually exist in some sense, nor does it propose reduction to some psychological basis. Rather, we follow the concepts and methods of members, warranted by member's competance[4], as recommended by ethnomethodology.

Semiotics [30, 36] emphasizes the separation and relatedness of signs and meaning, as well as (via Saussure) the structure of complex signs, though many proponents seem to be covertly Platonist. We have also drawn upon activity theory, with its emphases on historical development, cultural context, and material mediation [5, 41]. Cognitive linguistics [21, 8] provides the important notions of conceptual space and blending, and emphasizes the role of metaphor. Actor-network theory [23, 3] focuses on the numerous, diverse actors and complex relationships needed to sustain islands of stability in social experience, and also emphasizes the role of representations and translations. Discourse analysis [20] focuses on the structure of linguistic units larger than the sentence, and on certain relations in the coevolution of language and society; it also provides a window into the value systems of social groups. In particular, we use the notions of discourse saliency, discourse strength, and discourse type; the first two are discussed in Section 4 and the last in Section 7.

When we speak of the "social construction of mathematical objects," this should not be understood in a perjorative sense. Rather, as suggested by Harry Collins [6], it is a *methodological necessity* for a sociology of mathematics, as for the sociology of any science, to adopt a neutral, or even skeptical, stance towards what is studied; if we simply accept what practitioners tell us, we are not studying what they do. This is not to say that what mathematicians say is wrong, or that they simply make it up, but rather that methodological skepticism is essential to an empirical study, which must include studying what members say they do as a part of what they do[5].

[4]The author's professional training in mathematics includes a 1968 PhD from the University of California at Berkeley.

[5]This perspective underlines the fundamental misunderstanding on which some attacks on the social sciences in the so-called "science wars" have been based, e.g., [40], which (perhaps deliberately) confuses methodological skepticism with a denial of the validity of scientific results.

3 Some Literature

The literature on mathematics (as opposed to the literature of mathematics) is rather chaotic; much of it is speculative, normative, political and/or irrelevant, and relatively little is based on empirical studies of natural mathematics. I prefer not to cite examples from this depressing collection, but instead offer a rough typology for it. One large category consists of famous mathematicians pontificating, based on their experience with very hard proofs; like the advice of Olympic athletes, this may not be very useful to ordinary mortals, and could even be harmful. For a perhaps extreme example, as a graduate student, I was told by an eminent algebraicist that smoking was necessary for doing top quality algebra! Another, very large, category is written by philosophers, proposing theories of what is real. Educationists have also written much, sometimes based on classroom experience; these papers often exhibit a strange mixture of theories, and sometimes have hidden political agendas. Finally, psychologists have written about mathematics, often based on laboratory experiments in highly artificial environments.

We now consider some research closer to our own, in having an empirical basis in natural mathematics. A brilliant book by Lakoff and Núñez discusses the embodinent of mathematics [22]. Their toolkit from cognitive linguistics includes image schemas, conceptual metaphors, and blending, but not discourse, narrative, or sequential analyses, multimedia, or introduction and reference. There is much good material on the historical development of certain concepts, and on metaphorical projection, and there is an excellent analysis of mathematical concepts, which is not part of present study.

Eric Livingston wrote a classic ethnomethodological analysis of mathematics [26]. Its major example is an excellent discussion of Gödel's incompleteness theorem. Livingston finds (and we confirm) that proofs are constructed to be locally adequate for a given purpose, and in particular, are only elaborated to the extent needed for a particular occasion. Also, proving is *accountable* in the sense of making clear what it is; accountability is a natural social achievement of members. This work may be frustrating, because its strict adherance to ethnomethodology keeps it within the world of working mathematicians, and so prevents it from reaching what most people would call conclusions about this world.

Anna Sfard works on mathematics education, especially how new concepts are learned [38, 39], using a mixture of methods to study mathematical discourse, including semiotics and cultural psychology. Topics discussed include abstraction, learning, concept, metaphoric projection, and object; [38] anticipates some points from [22].

In his social study of mechanical theorem proving for verifying software, especially safety critical software, MacKenzie [27] discusses "cultures of proving," especially those of professional mathematicians, and of fully formal, mechanical proof in theoretical computer science. The present paper is focused on the culture of professional mathematicians, in which proofs are not formal, but we occasionally contrast this with the culture that advocates formalized proofs.

The present paper is also part of a larger project on what we call "natural ethics," the aim of which is to reveal inherent values in objects, by examining the work that is done in using them. For example, the values embedded in user interfaces are examined in [15], focusing on some popular web search engines. The paper [11] draws on ideas from ethnomethodology and the sociology of science, to develop foundations for a social semiotic approach to information, with examples from computer systems design, and to argue that values are inherent in all natural social interaction.

4 Object Introduction

Objects are introduced into mathematical discourse in a wide variety of ways, which writers and readers, speakers and listeners, tend to take for granted. But we will show that the different forms express the values of provers, including mathematical significance, mathematical difficulty, and what we will call ontological status (which is a kind of "degree of existence"), and we will show that these values are then used in resolving abbreviated references to objects. Our analyses rely on a principle called **recipient design** in ethnomethodology, which says that speakers tend to design what they say to minimize the effort required for listeners to understand it, taking advantage of shared knowledge and values. This implies that extra work done by speakers (or writers) generally has a purpose that relates to the work done by listeners (or readers). Although we do not have access to the values of writers (or speakers), we do have access to the work done by readers (or listeners) in interpreting what they write (or say), and so the principle of recipient design gives us access to certain work done by writers, and through that to their values. Since we can assume that writers consider this work worth doing, this allows to infer values shared by the community of professional mathematicians.

We also rely on some concepts and principles from discourse analysis, e.g., [20, 24], including the following: **Discourse strength** refers to the level of emphasis with which the object is introduced, **discourse saliency** refers to the likelihood that an abbreviated reference (such as "it") will indicate the object, and **discourse scope** refers to the area of text over which an introduction holds sway[6], i.e., the portion of text where the saliency of some object is high. We will see that discourse strength depends on a variety of factors, including lexical choice, syntactical placement, and discourse placement, and that there are some interesting relations among these discourse notions and certain values that are attached to objects. (In general, member's competance, i.e., proficiency in the area of mathematics involved, is needed to assess these factors, especially discourse scope, although these are analyst's concepts rather than member's concepts.)

Let's begin with some examples of simple imperative sentences that introduce objects into mathematical discourse:

Let N be an integer.
Assume that N is an integer.
Suppose N is an integer.

One purpose of introducing mathematical symbols like N above is so they can be used later; of course, they are tokens, not the mathematical objects themselves. It is easy to find many more instances of each of these forms, where "N" and "*integer*" are replaced by other tokens, such as "G" and "*group*," and of course many similar examples can be found in our data.

If mathematics is viewed as a purely formal activity, then the above introductions are equivalent; thus, in typical formalizations (e.g., in the type theory of Coq [7], or of Mizar [29]) they would all appear in the same form, as a declaration of a variable N of type `integer`. But natural mathematics is not a purely formal activity, it is a human activity, and natural mathematical discourse involves many subtle distinctions. For example, the "*let*" form above is stronger than the "*assume*" form, which in turn is stronger than the "*suppose*" form, in that these forms express progressively more doubt. Passive forms of introduction, such as "*Let a group G be given,*" are still weaker. We will see other examples of such hierarchies.

Sentences that begin with "*let*" are not very common[7] in ordinary English discourse, so it is interesting to consider why they are so common in mathematical discourse. A phrase of the form "*Let X be a Y*" perhaps takes a magisterial aura through association with one of the most famous phrases in the Bible, "*Let there be light,*" which is said by God, and clearly involves creation. The phrase "*Let it go*" is somewhat common in ordinary discourse, but since it does not involve the verb "*to be*" it has a different character. The phrase "*Let it be,*" which is the title of a famous Beatles song, is somewhat common in

[6] In computer science, the term **lexical scope** is used for the portions of a computer program's text within which a declaration is active. In contrast to natural texts, this notion has a completely precise definition for each programming language.

[7] However, sentences that begin with "*let's*" or "*let us*" are common, and can even be found in this paper.

New Age and other circles, and also seems not closely related. Sentences like "*Let the games begin*" or "*Let the good times roll*" are performative speech acts in the classic sense of Austin [1], in that they ritualistically call forth the condition described (the first of these begins the Olympic games). Getting closer to the usage in mathematics, we can imagine[8] someone saying "*Let T be the table*" as they go over a diagram showing the placement of furniture in a room, and maybe even "*Let T be a table*" as they draw it on the paper. Such uses are certainly related to those in mathematics, and highlight the baptismal aspect, since many uses of "*let*" give names to objects that (implicity) already "exist." Note that the implicit subject of the verb "*let*" is "*I*" or "*we*," where the latter has some connotation of the so-called "Royal we," where a monarch or other important person uses first person plural pronouns for him- or herself. In summary, phrases of the form "*Let X be a Y*" or "*Let X be Y*" confer a name, have an association with creation, and have a God-like or royal connotation, all of which help to make mathematical objects appear real. We will argue in Section 8 that what such locutions call forth is a conceptual space, in the precise sense of cognitive linguistics.

Although all the above introductions occur in main clauses, there are also many examples where introductions occur in a subordinate clause, which may come before or after the use of the object:

> *Given an integer N, ...*
> *Supposing that N is an integer, ...*
> *Assuming N is an integer, ...*
> *For N an integer, ...*
> *If N is an integer, ...*
> *..., where N is an integer.*
> *..., with N an integer.*
> *..., assuming N is an integer.*
> *..., provided N is an integer.*

A general principle in discourse analysis is that something placed in a main clause is stronger than something placed in a subordinate clause; more generally, the more deeply embedded the phrase in which something appears, the weaker it is. In addition, introductions that come before use are stronger than introductions that come after. Other forms of emphasis include repetition, italicization (or other font changes), color changes, and in spoken language, certain changes in volume, spacing, or intonation.

The scope of introductions tends to be much more carefully controlled in mathematical discourse than in ordinary discourse. For example, phrases like

> *Throughout this chapter, ...*
> *In this section, ...*
> *Within this proof, ...*
> *In this lemma only, ...*

are often used to qualify object introductions. See also the discussions of scope earlier in this section, and in Section 5 for assertions.

All the introductions above identify a signifier, such as N, with what is signified. Although this is the most common case, there are also many introductions that indicate a separation between signifier and signified, such as the following:

> *Let N denote an integer.*
> *..., where N indicates an integer.*
> *..., where N ranges over the integers.*

[8] These items are not in our dataset.

The identification of signifier and signified contributes to the appearance of the reality of the object involved, since the signifier is certainly real, e.g., ink on paper, chalk on a blackboard, or sound in the air. Therefore the above phrases have less strength than the more direct forms.

Objects can also be introduced with some assumptions, as in the following,

> *Let N be a positive integer.*
> *Suppose P is a prime greater than N.*
> *Assume that G is a commutative group.*

This is a slight extension of the syntactic form for basic introductions.

Objects are not always given a symbol when they are introduced:

> *We will operate over a fixed real closed field.*
> *We work in an arbitrary Hilbert space.*

Introductions with forms like this give their objects a larger scope and thus a higher saliency than those previously discussed, because it is clearly intended that they hold over a large segment of discourse. Note that such introductions could also include a symbol, as in

> *We will operate over a fixed real closed field R.*
> *We work in an arbitrary Hilbert space H.*

It is not necessary for objects to be introduced as specific individuals, since they can instead follow some general convention that has been explained earlier, such as:

> *We let capital letters denote sets.*
> *We will use the notation $\vec{a}, \vec{b}, \ldots$ for vectors.*

One can then write formulae using the symbols that these conventions justify without explicitly introducing the symbols. Introductions like these indicate a greater importance for the class of objects involved, but not necessarily for objects of the class, each of which should be judged from its own introduction.

For a mathematician, assertions and proofs are also objects that can be referred to, combined with other objects, and manipulated in various ways. So it should be no surprise that these are introduced into mathematical discourse in ways that are similar to other objects; we give examples and discuss the values that are attached to them in the next section. Actually, results and proofs are in general *much more* important in mathematics than objects, which after all are only introduced in order to facilitate stating results and constructing proofs. In particular, conditional assertions are extremely common, which helps to explain why introductions involving "*if,*" "*assume,*" and "*suppose*" are so common: they set up the conditions under which some assertion is to hold. Often these introductions appear in clauses that are subordinate to the clause of the assertion, though this is certainly not always the case.

In general, the stronger an introduction, the larger is its discourse scope, and the more important is the object involved. We have already seen some factors that effect discourse strength, including syntactic features like depth of nesting within phrases, and discourse features, such as placement at the front of a major unit; the next subsection will discuss how lexical choice can also be a factor.

Let us call the mutual correlation of discourse scope, discourse strength, and importance, the **principle of proportion**[9]. For example, a "*let*" introduction at the beginning of a proof is likely to hold

[9]Such principles are neither normative nor descriptive, but rather are **conventional**, in the sense that participants are aware of them, and make appropriate use of them, including orienting to exceptions; this gives a precise meaning to the phrase "honored in the breach." Note also that this asserts a mutual correlation between two factors, rather than a one-way causal relation.

throughout the proof (unless it is specifically revoked), and to introduce an important object. On the other hand, if an assertion in some proof involves an index i that ranges from 1 to n, then its introduction typically appears in a subordinate phrase before or after the assertion, rather than near the beginning of the proof, i.e., in a form like

Let $x_i \in S_i$ *for* $i = 1, ..., n$.

instead of a stronger form preceeding the assertion, such as

Let i *be an index ranging from 1 to* n.

which would imply a larger discourse scope (though the situation could be different if there are many formulae indexed over that same range). In many cases, the formula is in a subordinate clause, so that the introduction is two levels down, and thus quite weak[10]. The systematic use of such conventions, which parallel those of ordinary conversation, contributes to the appearance of reality for the objects involved.

Note that such a local introduction could locally "override" or supercede an introduction having a more global scope, so that an object can have a very high saliency within a limited scope created by a specific introduction. For example, in a text that uses complex numbers, i would refer to $\sqrt{-1}$ outside the limited scope of the first introduction for i as an index above. This implies that the scope of an introduction may be discontinuous, rather than a continuous segment of text. One might argue that instead both scopes are continuous segments that overlap, but this runs contrary to the fact that it is considered very bad form to have the same symbol refer to two distinct introductions within the same assertion.

Instead of being denoted by a single symbol, objects can be denoted by complex symbols, as in:

angle APB
the integer $a - b$
the quotient group G/H
$\sum_{i=1}^{n} i^2$
$\frac{\partial}{\partial y}\frac{\partial}{\partial x}\int_0^y \frac{e^{xz+1}}{x+z}dz$

We see that an accompanying descriptive phrase in English is optional, and that all of these except perhaps the first have at least a connotation of doing a construction, such a subtraction, quotienting, summing, differentiating, integrating, etc. Note also that these constructions can be almost arbitrarily complex (e.g., one finds formulae that take a whole page of text, or even more). However, these constructions may be very complex and abstract (as opposed to relatively simple concrete constructions like addition), and they may even rely on non-constructive existence theorems.

4.1 Ontological Status

The main activity of professional mathematicians is not exposition, but exploration, in which one often does not know whether or not certain objects exist. Phrases like

and so we find ...
thus we discover ...

reflect a metaphor of exploration, rather than of creation, even though they occur in exposition. However, creation metaphors often do occur in constructions, as for example in

[10] Note that this discussion relies heavily on syntactic structure.

We drop a perpindicular.
Now bisect angle APB, and extend the bisector ...

The language of construction is pervasive in traditional Euclidean geometry, but is also found elsewhere, for example, in the very common use of "*show*" as synonym for "*prove*," as in phrases like "*we now show that ...*". Although less common, "*demonstrate*" has a similar character.

The reader may notice that ...
Now we observe that ...
thus we can see that ...
if you look into it, you will see ...

Verbs like "*see,*" "*observe*" and "*notice*" are often used to introduce results, and have a flavor of construction, which gives them a greater reality, and hence a greater sense of importance. Objects that are introduced in constructions, or that are introduced using a rhetoric of construction, have a higher ontological status than those that are introduced in a more logical mode.

We argued above that objects introduced using verb forms like *suppose*, *assume*, and *granting* have a greater level of uncertainty than objects that are introduced using *let*, etc.; we will use the term **ontological status** to to refer to this particular value that is associated to objects. Objects with a greater ontological status have a greater certainty, and as a result have a greater appearance of reality.

Lower levels of ontological status often occur in proofs by contradiction, in which a thing assumed to perhaps exist is shown not to really exist. Here is a simple example:

We will show that there are no even primes greater than 2.
Suppose that p is such a number.
Then $p = 2n$ for some $n > 1$.
Therefore p is not prime.

Because such proofs can be very confusing to newcomers, it is worth considering how to make them easier to follow. Here is a version of the same argument with a more dramatic introduction, and also an explicit indication that the proof is over:

I claim there are no even primes greater than 2.
Suppose that p is such a number.
Then $p = 2n$ for some $n > 1$.
Therefore p is not prime.
QED

The rhetoric of truth and reality is used when a proof succeeds, i.e., "truth" is the mathematician's way of describing what happens happens in properly accountable proving. But this rhetoric breaks down when things are not going so well[11]. For example, phrases like

I'm not so sure about this, but if ...
Maybe we could just assume it and carry on.
Well, what if we try it this way: ...
Maybe there's some way to ...
I wonder if we really need that?

reflect uncertainty, and reduce the ontological status of any objects to which they refer. We have also found a strong correlation between the force of such expressions, and the importance of the result being undermined. For example, we see stronger language, including strong lexical items, in the following cases where an attempted proof has completely broken down:

[11]Unfortunately, we have much less data on such phenomena, so our conclusions are somewhat limited.

I just can't figure it out.
*Sh**, it doesn't work!*
I think it's false now.
No, that's not it.

Negatives seem to feature prominantly in the discourse of failure, where the ontological status is of course very low, or even zero when an assertion is proved not to hold, e.g., by giving a counter example.

5 Assertion Introduction

As already noted, assertions may be considered objects, and much of what we observed about objects applies to assertions; however, assertions are also treated differently from objects in some ways. For example, the most important results in a book or paper are often introduced in a separate paragraph, headed by keywords that include "*Theorem*," "*Proposition*," "*Lemma*," and "*Fact*." Note that these form a hierarchy which encodes the mathematical importance assigned to results. Assertions in this form are often numbered sequentially, as are definitions and sometimes other blocks of text. The numbering scheme may start over in each chapter of a book or section of a paper, and may include the chapter or section number, giving rise to forms like "*Theorem 7.4*" and "*Definition 2.1*," instead of "*Theorem 27*" and "*Lemma 18*." More elaborate forms are also possible, such as "*Proposition 8.6.3*." In books and papers, formulae are often enumerated using forms like

$$[6.21] \qquad e^{\pi i} = -1 \ .$$

In oral mathematics, simple numbering schemes may be used for sets of rules, axioms, and similar homogeneous collections, but more elaborate schemes, like those just described, do not appear.

Very important results are often given proper names, e.g., "*the Prime Number Theorem*," "*Zorn's Lemma*," "*Hilbert's Nullstellensatz*," etc., and these appear in both written and oral discourse. As with other objects, formulae can also be given short names when they are introduced,

Let F be the following formula, ...
Let F be the above formula.
Let F denote the following formula, ...
Let F be the formula $\underline{.....}$

where we use $\underline{.....}$ to indicate a gesture, such as pointing. (This is our first example of notation for multimedia discourse, which in general is very difficult to notate with precision as great as that of purely linguistic discourse.) The considerations discussed in Section 4 also apply here, supporting notions of discourse strength and ontological status for assertions, though it is clearer to speak of a "truth status" for assertions, than of an ontological status. The discourse strength of assertions is determined in much the same way as for objects, mainly on the basis of syntactic and discourse structure. A simple example appears in the proofs by contradiction in Section 4.1, which each begin with an assertion introduction; here the second introduction is stronger, due to its stronger syntactic form.

Assertions are also given another value by mathematicians, which measures how difficult they are to prove. This is not only a very basic (though somewhat covert) value in the mathematics community, but it also provides useful information for proof readers, because it can help in knowing when the proof of some assertion is finished, and more generally, it can help in navigating proof structure, which is often difficult for complex proofs.

Inside a proof, an assertion may be "open" or "closed," which respectively mean that it is in the process of being proved, or that it has been proved and is available for use in proving other assertions; we will call this the **proof status** of an assertion. The situation is actually more complex, because an assertion may be used before it is proved, provided that its status is made clear, that it really is proved later on, and that no circularities are involved[12]. Moreover, in proofs that are still under construction (i.e., in mathematical research), the status of many assertions is often somewhere between open and closed. This implies that we need several different notions for assertions that are analoguous to discourse scope for objects. One of these is the portion of text within which an assertion is closed, or is treated as closed; another is the portion of text within which an assertion is open; another is the portion of text within which an assertion is undergoing proof; and still another is the portion of text within which an assertion is salient at all, independent of its proof status. Let us call these the **closed scope**, **open scope**, **proof scope**, and **salient scope** of an assertion. For a given assertion, its salient scope will contain each of the other three. These notions can help us understand how proofs are navigated, noting that navigation can be difficult for complex proofs.

The following further examples of assertion introduction are ordered by their discourse strength,

We claim that ...
We will prove that ...
It follows that ...
Observe that ... *Notice that ...* *Note that ...*
It can be shown that ...

The first two introduce new open assertions, whereas the remainder introduce assertions that are to be considered closed (either through a subsequent short justification, or through a missing justification that the reader is expected to supply). All of these are stronger than the following single word introductions,

Therefore, ... *Then, ...* *Thus, ...* *Hence, ...* *So ....*

which each introduce new closed assertions (or assertions that are regarded as closable by the reader), and which again are ordered by their strength.

It is also of course possible to introduce assertions in subordinate phrases, for example, beginning with words like the following, which again are ordered by strength,

if, where, when, provided, granting

in that these words suggest progressively more doubt, although this is used to express their discourse status rather than their truth status. The words in the list "*therefore*," "*then*," etc. above can also be used to introduce assertions in subordinate clauses; these have a lesser discourse strength than those in the list "*if*," "*where*," etc.

It is well known to professional mathematicians that phrases like

It is clear that ...
One can easily see that ...
Obviously ... *Certainly ...* *Evidently ...*
A little calculation shows that ...
The reader can [easily] check that ...

often hide a tedious or tricky calculation. This convention can be very irritating to those outside that narrow circle, inlcuding students. Conventions like this of course serve to solidify the boundaries of a professional group, but they also reveal its values, in this case, a negative value that is placed on calculcation, with a corresponding positive value placed on more creative aspects of proofs. This is quite opposite to the values of most non-professionals, who would in general prefer easy proofs, and would rather see more details of calculations.

[12] So called "natural deduction" proofs enforce the discipline of proving assertions before they are used; but in fact, this is unnatural, and is often violated in natural mathematics.

There is a **principle of proportion** for assertions analoguous to that for objects, asserting the mutual correlation of salient scope, discourse strength, and mathematical importance. Again, this principle is neither normative nor descriptive, but rather is a convention, to which members orient, including in cases where it does not hold (see footnote 9). For example, named results are stronger and therefore assumed to be more important and to have larger scope than unnamed results. An example from our data is discussed at the end of the next section.

6 Reference

Mathematical discourse builds on ordinary discourse, and in particular, it builds on the conventions of ordinary discourse, including the modes of reference that are employed in various discourse types. Ordinary discourse presupposes concrete objects, and requires the listener (or reader) to determine the concrete referents of words and phrases such as

it, that, this,
that one, the other, the last one

All these phrases occur in mathematical discourse, but mathematical terms may also occur, as in

this formula, that integer,
the other variable, the previous lemma

In addition, mathematical discourse has some more specialized conventions for reference. For example, if a formula has been named "[6.21]", then that name can be used to refer to the formula later, or even earlier, in the text; an abbreviated form like "*formula* 21" can also be used within the scope of "6," which could be a section of a paper or a chapter of a book. Similar reference conventions apply to theorems, definitions, and whatever else has been numbered in a similar way in a text. These conventions refer to the large-grain structure of texts, rather than the structure of proofs, except insofar as this might be reflected in the text structure.

The extreme rarity of such precise forms of reference in ordinary discourse compared to their relative frequency in mathematical discourse reveals the importance that mathematicians attach to precise reference to mathematical objects, as well as to their great sensitivity to scope in mathematical discourse.

It is interesting to notice that chains of reference can also occur, as in

the formula that we used to prove this one
the proof of the previous lemma
the second integral in the third formula below

It is important to notice that the values attached to mathematical objects are used in resolving later references to those objects in mathematical discourse. For example, phrases like

The proof is now reduced to ...
The desired result now follows.

require us determine which "*proof*" and which "*desired result*" are meant; their referent will be whatever unfinished proof or open assertion has the greatest current discourse salience, which is in part determined by their mathematical difficulty and significance.

Introduction can be rather complex in multimedia discourse:

Consider <u>this formula</u>.
Therefore <u>...</u>
What we want to prove is <u>...</u>
The last formula with x replaced by y.
The last formula with <u>this</u> substituted for <u>x</u>.
<u>This formula</u>, with <u>this</u> and <u>that</u> reversed.

where the underlines indicate a gesture, writing on the board, writing plus gesture, etc.; speech and writing and/or gesture can be simultaneous in forms like these. Also, values are often indicated in complex ways using body language. Similar phenomena occur when assertions are referenced:

Let's do <u>this</u> one first.
So all we need now is to prove <u>that</u> one.
The formula that used to be <u>here</u>.

The sequential organization of discourse has perhaps been studied with the greatest care and precision in the branch of ethnomethodology known as conversation analysis [35]. In particular, conversation analysis has studied introduction, reference and re-reference. One general principle is that references are constructed to take the least effort and still be effective [37], so that differences from such optimal phrasings must be doing some work in addition to reference, such as imparting additional emphasis, or recalling shared values (see footnote 9). For example, the phrase "*by Cayley's theorem (Theorem 18)*" gives two different references to the same assertion, one of which is a proper name; this redundancy gives extra strength to the reference, and therefore the principle of proportion leads us to assume that this particular use of Cayley's theorem is important. Another result is that references to an object can be simpler for each of a series of references, because the saliency of the object rises the more it is referenced [37], as well as closer to a previous reference. For example, the first reference to a low salience object might be rather elaborate, but soon after that, it can be referenced using just "*it*" or "*that*". Conversation analysis has also studied the repair conventions used in ordinary conversation (i.e., the ways of correcting speech on the fly), and it would be interesting to see how these apply to the correction of mathematical speech.

6.1 Abstraction

We can get deeper into the nature of mathematical objects by examining the nature of mathematical abstraction. Nearly all mathematical objects are **abstract** in the sense that there are multiple representations, which are considered to be "the same," or more formally, equivalent, as in the following, where each line contains different terms, which are considered "equal" in some sense:

$$\begin{array}{lcllcl}
[10,000,000]_2 & = & [1024]_{10} & & & \\
\frac{2}{6} & = & \frac{1}{3} & & & \\
.99999... & = & 1 & & = & 1.0 \\
0+5 & = & 5 & & & \\
x \times y & = & y \times x & & &
\end{array}$$

Each notion of equality is different. The first concerns the representation of an integer in base 2 (binary) and base 10 (decimal). The second concerns the equality of fractions, while the third line concerns real numbers. The last two concern the results of operations on numbers. In each case, an equivalence relation is represented by the equality sign, which asserts that the relation holds between two expressions. This is very typical in mathematics, and serves to reinforce the appearance of existence of an "ideal" entity which is "represented by" all the terms. Plato's famous cave, in which only the shadows of "real objects" can be seen, but not the objects themselves, seems an almost perfect metaphor for this situation, including the fact that there is no evidence for existence of the "real objects" themselves[13].

[13] Contrary to what many people think, even in pure mathematics there is no such thing as the ideal number one itself. Rather, the number one is constructed or represented in various ways in different contexts. For example, in set theory, it may be constructed as the set containing the empty set, "$\{\emptyset\}$" (there are also other constructions). In Peano's axiomatization of the natural numbers, it appears as the term $s(0)$. Among the fractions, it appears as the set of pairs of integers (a, b) with $a = b \neq 0$. The latter is an instance of a common trick, which is to "divide by" the equivalence relation that underlies the equality sign used for some particular class of representations.

It seems that abstraction, in the sense of (often implicit) equivalence relation, may be the essence of mathematical objects.

The work of calculation often consists of producing a sequence of different representations connected by the equality sign, as in the following:

$$\begin{aligned} \tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4} &= \\ \tfrac{6}{12}+\tfrac{4}{12}+\tfrac{3}{12} &= \\ \tfrac{13}{12} &= 1\tfrac{1}{12} \end{aligned}$$

Even though the aim is to reach the last term, one tends to think that every term "represents the same thing," i.e., is a reference to the same "abstract" mathematical object.

There is another sense of the word abstract, according to which a number of objects are regarded as "instances" of a single "more abstract" object (which itself is likely to be abstract in our first sense); this also can be seen as a relation, though not in general an equivalence relation. For example the equations $x+y=y+x$ and $u+v=v+u$ are equivalent[14], the abstract object here is "the commutative equation", and the equations $4+5=5+4$ and $8+11=11+8$ are two instances of it, obtained by substituting particular integers for the variables.

7 Proof Structure and Narrative Structure

We first explain the important notion of **discourse type**. This is a collection of conventions (in the sense of footnote 9) for organizing and interpreting certain units of language. These conventions include ways of opening and closing a unit, and for connecting clauses within a unit. Usually there is a primary type of clause and a **default connective** for such clauses, which can be omitted without changing the interpretation, and which represents the most important way to connect clauses in this discourse type. There may also be other conventions. A **discourse unit** is an instance of a discourse type, e.g., a particular text, such as the small proof by contradiction in Section 4.1. The proof and narrative discourse types are discussed below. In [33], Sacks already noted that narrative structure is governed by conventions, rather than rigid rules as in Chomskian formal grammar. Narrative is arguably the most basic of all discourse types. Other discourse types include plans [25], explanations [18], jokes [34], and command and control [17].

Our primary interest in this paper is the proof discourse type; it is more complex than other types that have been studied. Proofs open with a formal announcement of what is to be proved, marked for example by "**Theorem:**", or in a sentence such as "*We will prove that ...*". Proofs are typically closed with a formal symbol such as "□" or "*QED*", or with a sentence such as "*And thus we are done*" or "*And that concludes the proof.*" There are two kinds of clause, for proof steps and for introductions. The default connective for proof steps is "*therefore,*" "*thus,*" "*hence,*" or some other synonym; but unlike most discourse types, this default connective is not usually omitted. However, the default connective for introductions, "*and,*" is often omitted. Other conventions for the proof discourse type have been discussed earlier in this paper, including ways to introduce, name, and reference objects. There are also conventions for indicating logical dependencies among proof parts, but we do not investigate them in this paper.

Proofs are presented linearly, i.e., as a sequence of steps, even though the underlying mathematical structure is often non-linear; the same is true of stories, which typically give a sequential account of concurrent interacting events and persons, through the use of interleaving threads; there may also be

[14] The relevant relation says that there is a one-to-one correspondence between their variables, i.e., they are "the same" up to renaming of their variables.

flashbacks and flashforwards. A classic structural theory for the narrative discourse type has been given William Labov [20] (refined by Charlotte Linde [24]), saying that stories have:

1. an optional *orientation section* at the beginning;
2. a sequence of *narrative clauses* for actions, which are interleaved with
3. *evaluative material* giving reasons for actions; and
4. an optional *closing section.*

The orientation section may set the time and place, and introduce some characters. For example, fairy tales prototypically begin with "*Once upon a time.*" The narrative clauses are generally in the narrative past tense, and the events that they describe are presumed (unless otherwise indicated) to occur in the order that they are presented; this principle is called the **narrative presupposition**. For example, in "*he came, he saw, he conquered,*" the narrative presupposition tells us that these three clauses refer to events that occurred in the indicated order. The default connective here is "*and*" or "*then,*" because these can be substututed for the missing connective without changing the meaning, giving "*he came, and he saw, and he conquered,*" or "*he came, then he saw, then he conquered.*" The form without connectives is more usual, and the use of either of the other two forms would indicate some additional emphasis. Evaluative material may appear in a dependent clause, or in unexpected syntactic or lexical choices, such as repetition or swear words; evaluative material is a rich resource for uncovering values. Narrative structure serves to support the appearance of reality for mathematical objects, because it is a familiar form that ordinarily involves concrete objects and events.

The discourse structure of proof appears to be derived from that of narrative, because echoes of narrative structure are found in proofs. For example, the primary default connective in proofs is "*then*" (with its synonyms like "*thus*" and "*so*"), although with the meaning of logical implication rather than temporal succession. That "*then*" has originally a temporal character is shown by its appearance in phrases like "*now and then*" and "*and then again...*", and "*then*" can also be used in proofs to indicate temporal ordering, for example in phrases like "*then we will show that...*" In addition, words like "*next*" and "*and*" are also used in proofs to indicate sequential ordering. Although the primary default connective can be omitted, it sounds awkward. But as in narrative, temporal connectives can be omitted in proofs, which explains why it is unusual to omit the logical implication connectives. Notice that the conjunction "*and*" also has both a temporal and a logical meaning, exactly parallel to "*then*". We can illustrate these ideas with modified versions of the proof by contradiction from Section 4.1, first a discourse unit with no connectives:

> *Suppose* p *is an even prime greater than 2.*
> $p = 2n$ *for some* $n > 1$.
> p *is not prime.*
> *QED*

Now let's see it with "*and*":

> p *is an even prime greater than 2, and*
> $p = 2n$ *for some* $n > 1$, *and*
> p *is not prime.*
> *QED*

Clearly, the proof sounds better using "*then*" and its synonyms:

> *Suppose* p *is an even prime greater than 2.*
> *Then* $p = 2n$ *for some* $n > 1$.
> *So* p *is not prime.*
> *QED*

Notice that understanding the first of the three proofs above involves an analog of the narrative presupposition, because we naturally presuppose that each clause follows from what comes before it; this highlights the basic importance of inference in proofs. However, proofs can also use a converse presupposition, in which the default connective is "*because*" or "*since*" or a synonym, instead of "*then.*"

The use of the temporal connective "*then*" for implication in proofs is part of a general pattern, other examples of which are "*it follows that ...*" and "*after which ...*". This is an instance of what George Lakoff and Rafael Núñez [22] call a conceptual metaphor; it maps from an image schema for temporal succession into the more abstract domain of logic (the notions of image schema and metaphor are discussed in Section 8 below). Let us call this the LOGICAL CONSEQUENCE IS TEMPORAL SUCCESSION metaphor. Notice that the logical meaning of "*since*" in proofs is also part of this pattern.

A common complaint by non-mathematicians about proofs is that their structure is not clear. For example, a lemma may be stated well before its role is (or can be) made clear, or a seemingly unrelated claim may be proved before drawing a surprisingly short proof of the main result from it. Mathematicians enjoy these sorts of tricks, but outsiders do not usually share their pleasure. This brings out a further value of the community of professional mathematicians, which is to be clever.

We have already noted that traces of narrative occur in proofs. Another research project has explored the hypothesis that a careful use of narrative structure can make proofs easier to follow. One inspiration has been a video tape of an explanation of the operation of a mechanical theorem prover, in the form of an epic narrative of proof attempts by the program; it is rather like the narratives described by Joseph Campbell [4], in which a hero overcomes a series of obstacles. The hypothesis has been tested (anecdotally, not statistically) in a web-based proof display system that we have developed [16]. Each proof has a homepage which serves as an orientation section; evaluative material (with motivation, explanations, etc.) is interleaved with proof steps, which are presented in the converse order (unless the prover specifies otherwise); there is also an optional closing page, which can summarize what has been proved and what can be learned from the proof [13]. Feedback received from students and non-professionals has been very positive, although professional mathematicians often consider the problem of improved exposition to be uninteresting.

Much more could be said on the topics of this section, for example, about proof navigation, but this would distract from the core of what we wish to say, and so is better left for a future paper.

8 Cognitive Linguistics

In their pioneering book [22], George Lakoff and Rafael Núñez demonstrate in detail how mathematics is grounded in everyday experience; in particular, they demonstrate that the language of mathematics contains many metaphors that are based on image schemas. Before explaining these notions, we first explain a foundational notion, that of conceptual space. The original notion (e.g., in [8]) says that a conceptual space consists of some objects and some relations among them[15]; to this, I would now add the concepts and methods, in the sense of ethnomethodology, that are relevant to these objects, and sometimes an evolving local state, as is needed for example by geometrical constructions[16]. Whenever a mathematical object is introduced, the conceptual space associated with objects of that type comes into the discourse with it. For example, the introduction "*Let G be a group*" imports group theory into the discourse (or at least whatever group theory is known and relevant at this point). Among things imported might be the concept of a subgroup, and methods for proving things about groups, for example, by using the Sylow theorems.

[15]As noted in [12], this makes conceptual spaces a simple special case of first order theories in the sense of mathematical logic; this observation suggests generalizing the notion to include constuctors and types for terms, as was done in [12].

[16]This is similar to object oriented programming in computer science.

An **image schema** is a conceptual space that arises from some innate sensory-motor schema, i.e., from human perception and coordinated action in the world[17]. One example is the CONTAINER image schema, which involves an inside, an outside, a boundary, and the possibility of things being inside or outside. Image schemas have a perceptual basis, and an associated conceptual space; a particular instance of a schema, arising when it is used, may be further enriched.

A **metaphor** is a mapping from a source conceptual space to a target conceptual space. For example, the SETS ARE CONTAINERS metaphor, which underlies much of elementary set theory, maps from the (conceptual space of the) CONTAINER image schema to (a conceptual space for) sets; this map takes things inside a container to elements of a set. Another mathematical metaphor is NUMBERS ARE POINTS ON A LINE, which maps from the (conceptual space of the) image schema POINTS ON A LINE to a conceptual space for the real numbers. Mathematical sentences understood using this metaphor include "*x is between 5 and 6,*" "*x is below 12,*" "*y is close to x,*" and "*x is far from 0.*" The phrase **metaphoric projection** (of or from S) is often used, where S is the source of a metaphor.

The use of metaphors based on image schemas supports the reality of mathematical objects in the target spaces, by relating them to our experience of being in the world. This observation helps to explain the applicability of mathematics: Sensory-motor schemas work in the world because they were selected over millions of years of evolution; therefore language that is based on them also works in the world. From this, we see that the apparent mystery of the applicability of mathematics depends on our accepting that mathematics is transcendental, and it disappears once we realize that mathematics has been grounded in real world experience from the very beginning. Several other arguments against the traditional transcendental view of mathematics may be found in [22]. Most mathematicians probably realize that the reality of mathematical objects is an illusion, but they are trained not to admit it, and may even be told that believing in this illusion will help them do better mathematics!

Lakoff and Núñez also demonstrate that conceptual blends, and in particular, blends of metaphors, play a very important role in mathematics, but we will not discuss this here; see [12] for a category theoretic treatment of blending. The book [22] contains many further concepts and interesting examples, and the reader is encouraged to consult it.

9 In Conclusion

This paper reports an empirical study of mathematics as social activity. Our data is natural mathematics, in the sense of actually observable proof events. A number of results appear to be new. Proofs have been proposed as a new discourse type, and aspects of natural proof structure have been described, including the introduction of, and reference to, mathematical objects. A large range of linguistic devices have been shown to reinforce the appearance of reality for mathematical objects, including the use of metaphoric projection from image schemas, and the metaphoric projection of the narrative presupposition to entailment. We have shown that the reality of mathematical objects is constructed and sustained by work done in discourse; it is not a self-evident objective fact. This work includes the use of conventions from ordinary discourse, which presume ordinary concrete objects, as well as object introduction and reference, narrative, image schemas, metaphors, blends, and abstraction.

Another kind of work done in mathematical discourse is the finely nuanced assignment of values to objects; these values are then used in resolving discourse references to mathematical objects. We have found evidence for the values of mathematical significance, difficulty, and ontological status. It seems

[17] This definition differs from that of Lakoff and Núñez in making the conceptual space explicit, and thus emphasizing that it is a *model* of a natural phenomenon constructed by analysts, rather than the phenomenon itself. This seems appropriate because conceptual spaces cannot be directly apprehended, and therefore must be inferred, and it has the advantage of avoiding the appearance of covert Platonistism.

that the work done to make values invisible is exactly what renders them visible to analysts.

Our analysis of values and work in proofs provides a basis for infering the values of provers, that is, of the community of professional mathematicians. These values include clarity of reference, clarity of organization, difficulty of proof, mathematical significance, concision, and surprise. That clear reference is important is evident from the unusual, multiple, precise mechanisms for reference that we have documented in this paper. Similarly, the importance of organization is inferred from the mechanisms used to support it in proofs, though we have not explored this as thoroughly as we did reference. However, we have discussed one important mechanism, the way that the ontological status value assigned to assertions helps readers understand proof organization. Difficulty of proof is a major value attached to assertions, as is mathematical significance. The importance of concision, and even austerity, can be seen in the extensive use of symbols to denote objects, as well as in the paucity of explanation. We have not previously argued for the importance of surprise, but it is easily seen in the way that many proofs end, sometimes "pulling a rabbit out of a hat," and sometimes leaving the reader to put things together after an abrupt "*QED*."

Our results also support some conclusions about the philosophy of mathematics. First, Platonism is false, in the sense that natural mathematics is always materially mediated, embodied, and situated. Although this does not exclude the possibility that Platonic objects have some reality of their own, it does exclude their having any material embodiment or practical effect, so that it is impossible to observe any causal role for ideal Platonic objects in natural mathematical discourse. On the other hand, Platonism is true, in the sense that natural mathematical discourse, and mathematicians, treat mathematical objects as real, and moreover, mathematical objects also presumably correspond to real events in the brains of mathematicians.

Formalism is the view that mathematics is just the manipulation of symbols according to certain rules, with no inherent meaning at all. This is false, because natural mathematical discourse is very informal, value laden, very expressive (even dramatic), and grounded in real experience (the last assertion is extensively argued in [22], based on image schemas, metaphors, etc.).

The results of this paper have significant implications for mathematics education. First, we have seen that natural mathematics is not routine: natural proofs are "site specific," and in particular, are only elaborated to the extent needed at a particular time and place. This implies that extreme forms of the "back to basics" movement, which rely exclusively on memorization, routinization, etc., are misguided. On the other hand, grounding everything in practical experience is also misguided, because it does not allow for abstraction, which we have seen is one of the most fundamental features of mathematics. Finally, since the philosophical position of formalism is wrong, teaching mathematics as pure formality is also misguided; it leaves out the vital life force of mathematics.

Although not directly supported by the data used in this study, it seems likely that similar points could be made about the application of mathematics to other fields, as well as about fields that rely extensively on mathematics, such as physics, chemistry, and engineering. For example, concepts like force do not have an actual Platonic existence, but rather are parts of complex models that have been constructed by physicists. Also, careful readings of physics texts will no doubt reveal values of the physics community, and it is likely that these values will be similar in many ways to those of the mathematics community, though different in other ways.

This paper is an early attempt to apply empirical methods (discourse analysis, ethnomethodology, and cognitive linguistics) to natural mathematics, so some gaps, omissions, and even errors are to be expected; we hope that this will inspire further work, since there is much more to be done, and there are important applications, not the least of which is secondary mathematics education, which is currently plagued by highly politicized polarities.

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